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ON A NEW NUMERICAL SCHEME FOR ICE DYNAMICS MODELS

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ABSTRACT

Recent anisotropic ice dynamics models have emphasized that velocity can be discontinuous across active leads, rafts, or ridges and have described these discontinuities explicitly. Here we use the fact that velocity discontinuities must be aligned with characteristic directions (slip lines) in these models. The analysis is limited to quasi-steady behavior, where time is a parameter in the mechanical behavior (no time derivatives in the momentum balance or constitutive law). The quasi-steady model must neglect tidal oscillations as well as inertia in the momentum equations for ice and upper ocean and resolve time steps to at least a day.

The spatial derivatives are written in characteristic coordinates, a nontrivial transformation to non-orthogonal, curvilinear coordinates. We assume that each nodal point can be split into four or two nodes if the solution is hyperbolic or parabolic there. We have not yet determined which basic variables must appear in the equations, but expect that velocity and traction will appear. Since the characteristic directions must be determined as part of the solution, and the logical connectivity of nodes in characteristic coordinates is very difficult, it is not desirable to introduce a grid to describe the solution field. Therefore, meshless solutions methods will be investigated. We expect the characteristic solution method will provide a better description of small-scale ice behavior to better support offshore operations and shipping because of the explicit description of leads, rafts, and ridges. This new numerical approach will also allow comparison with other recent model developments.

INTRODUCTION

Modern anisotropic plasticity ice dynamics models allow us to describe explicitly the formation and evolution of leads, rafts and ridges. These linear kinematic features are important parts of the icescape, and a careful description of their presence and orientation can provide offshore operators valuable information on ice conditions. In this paper we present a different concept for an ice dynamics model and numerical solution method that describe the formation and evolution of leads, rafts, and ridges explicitly. The method is limited to quasi-steady models, for which we resolve temporal changes to a day or longer. Thus, at each step, time does not appear in the momentum equation or constitutive law. Here are the broad ideas.

- We integrate along characteristic directions, so that discontinuities appear naturally.
- We avoid determining the field of characteristic lines by making the scheme gridless.

- We allow each velocity node to be split into four or two velocity vectors when the equations are hyperbolic or parabolic, respectively.

To date, we have a concept for the new meshless model based on characteristics, we have an anisotropic constitutive law that describes lead formation and evolution, and we have derived the governing equations in characteristic coordinates. We know how the mass, momentum, moment of momentum and energy balance equations are expressed, and how the constitutive law relates stress and deformation. However, we do not know which fundamental variables are to be solved by these equations. Neither do we know where variables will be defined nor how nodes will be connected. In the following we present some useful first steps, but we have not yet developed a coherent structure that will lead to a new numerical solution method.

ANISOTROPIC PLASTICITY MODEL

The essential parts of the model are described in separate sub-sections here.

Momentum Balance

The quasi-steady momentum balance equation equates the Coriolis, and tilt accelerations to the applied air and water stress and the divergence of internal stress

$$m\left(f\mathbf{B}_{\pi/2} \cdot \mathbf{v} + g\nabla H\right) = \boldsymbol{\tau}_a - \boldsymbol{\tau}_w + \nabla \cdot \boldsymbol{\sigma} \qquad (1)$$

where m is area mass density, f is the Coriolis coefficient, $\mathbf{B}_{\pi/2}$ is a 90 deg counterclockwise rotation (ccw), $\mathbf{v}$ is ice velocity, g is the gravitational constant, H is sea surface elevation, and $\boldsymbol{\sigma}$ is internal ice stress. Air stress $\boldsymbol{\tau}_a$ is applied as a known quadratic function of geostrophic or surface winds and water stress $\boldsymbol{\tau}_w$ is a function of ice velocity relative to ocean current.

Constitutive Law

The anisotropic plasticity constitutive law describes explicitly when a lead or ridge forms and how it evolves (*Coon, et al.,* 1992, 1998; *Pritchard*, 1998). This is the fundamental difference between the anisotropic and the isotropic laws. When a lead is open, a uniaxial stress state must exist. The traction (normal stress and shear stress) across the lead must be zero, but the thicker surrounding (or neighboring) ice can support stress along the lead direction. This is perhaps the biggest difference between the anisotropic plasticity constitutive law and the isotropic laws.

The direction of each such feature must be retained as part of the solution. Since behavior can differ in different directions the constitutive law must be expressed in the three stress components ($\sigma_{xx}, \sigma_{xy}, \sigma_{yy}$), rather than the stress invariants used for isotropic laws. An anisotropic elastic-plastic constitutive law has been described by *Pritchard* (1998, 2008). Therefore, we present only enough detail to derive the governing characteristic equations.

The yield constraint is assumed in the general form to satisfy

$$\phi(\boldsymbol{\sigma}, \mathbf{K}) \leq 0 \qquad (2)$$

where $\mathbf{K}$ is a tensor containing strength parameters for ice having each different orientation.

Plastic flow is assumed to be normal to a potential function so that

$$\mathbf{D}_p = \frac{\partial \psi}{\partial \boldsymbol{\sigma}} \qquad (3)$$

where ψ is a potential function, and $\psi = \phi$ if a normal flow rule is assumed. It is quite common in this constitutive law for stress to lie on an intersection of yield surfaces defined for features having two different orientations., In this case, the plastic stretching is composed of contributions from each of the deforming features

$$\mathbf{D}_p = \sum_{j=1}^{m} \lambda_j \mathbf{n}_j \tag{4}$$

where $\mathbf{n}_j$ is orthogonal to the potential surface ψ_j, and m is the number of branches undergoing plastic deformation. The multipliers λ_j must be nonnegative. The normal tensor to each branch of the yield surface is

$$\mathbf{n}_j = \frac{\partial \psi_j}{\partial \boldsymbol{\sigma}}. \tag{5}$$

where ψ_j are potential functions and $\psi_j = \phi_j$ for normal flow rules. Plastic stretching of the j-th lead controls its redistribution.

For completeness, we define constitutive behavior when stress is within the yield surface. The details are not important, however, so we write simply that

$$\boldsymbol{\sigma} = \boldsymbol{\Sigma}(\mathbf{D}). \tag{6}$$

where plastic stretching $\mathbf{D}_p = \mathbf{D}$ equals total stretching and $\boldsymbol{\Sigma}$ indicates a general tensor relationship for a viscous closure law.

MATHEMATICAL CHARACTERISTICS

Pritchard (2008) derived the mathematical characteristic drections for this quasi-static model. Results are comparable to *Erlinggson* (1988), *Pritchard* (1988), and *Collins* (1989) who analyzed isotropic models. The velocity vector can be discontinuous across a characteristic line. The stress tensor can also be discontinuous, but the traction vector $\mathbf{t} = \boldsymbol{\sigma} \cdot \mathbf{n}$, where $\mathbf{n}$ is the unit vector orthogonal to the lead, must be continuous. This restriction allows the normal stress component along the lead to be discontinuous across the lead.

Here we derive the governing equations expressed in characteristic coordinates. When stress is on a smooth part of the yield surface, the system of equations has six variables: three stress components $(\sigma_{xx}, \sigma_{yy}, \sigma_{xy})$, two velocity components (u, v), and the scalar multiplier λ. These variables must satisfy two momentum equations, three flow rule equations, and the yield constraint. The case where two yield surfaces intersect can be included by assuming that small fillet radii smooth the transitions.

Stress characteristics

The highest order derivatives of stress appear in the momentum equations (e.g., *Pritchard*, 2008)

$$\begin{aligned}
&\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \tau_{ax} - \tau_{wx} + mf\left(v - v_g\right) = 0 \\
&\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial x} + \tau_{ay} - \tau_{wy} - mf\left(u - u_g\right) = 0
\end{aligned} \tag{7}$$

where (u_g, v_g) is the geostrophic ocean current, air stress (τ_{ax}, τ_{ay}) is a quadratic funcrion of the wind, and water stress (τ_{wx}, τ_{wy}) is a function of the relative velocity $(u-u_g, v-v_g)$. The stress characteristic analysis is performed by affixing the stress components into a two-component solution vector (shown here as the transpose)

$$\mathbf{Z}_\sigma{}^t = \left\{ \sigma_{xx} \quad \sigma_{yy} \right\}. \tag{8}$$

The shear stress σ_{xy} component can be determined directly from the yield constraint **(2).**

Two new independent coordinates ξ and η are introduced, with coordinate mappings

$$\begin{aligned} x &= X(\xi,\eta) \\ y &= Y(\xi,\eta) \end{aligned} . \tag{9}$$

We require that along each such direction, the governing equations have a derivative only in that coordinate, not the other. Derivatives of $\mathbf{Z}_\sigma$ along Cartesian coordinates are related to derivatives along characteristic directions by

$$\begin{aligned} J\frac{\partial \mathbf{Z}_\sigma}{\partial x} &= Y_\eta \frac{\partial \mathbf{Z}_\sigma}{\partial \xi} - Y_\xi \frac{\partial \mathbf{Z}_\sigma}{\partial \eta} \\ J\frac{\partial \mathbf{Z}_\sigma}{\partial y} &= -X_\eta \frac{\partial \mathbf{Z}_\sigma}{\partial \xi} + X_\xi \frac{\partial \mathbf{Z}_\sigma}{\partial \eta} \end{aligned} \tag{10}$$

where the Jacobian $J = X_\xi Y_\eta - X_\eta Y_\xi$.

The shear stress terms in equation (7) can be eliminated by differentiating the yield constraint along the characteristic directions and solving for $\partial\sigma_{xy}/\partial\xi$ and for $\partial\sigma_{xy}/\partial\eta$. Since there may be two such characteristic coordinates if the system is hyperbolic, there will actually be two such constraints. Along the ξ coordinate direction, its derivative expands to the form

$$\frac{\partial\phi}{\partial\xi} = \frac{\partial\phi}{\partial\sigma_{xx}}\frac{\partial\sigma_{xx}}{\partial\xi} + \frac{\partial\phi}{\partial\sigma_{yy}}\frac{\partial\sigma_{yy}}{\partial\xi} + 2\frac{\partial\phi}{\partial\sigma_{xy}}\frac{\partial\sigma_{xy}}{\partial\xi} + \frac{\partial\phi}{\partial\mathbf{K}} : \frac{\partial\mathbf{K}}{\partial\xi} = 0 \tag{11}$$

where **:** is the double inner product and the material parameter tensor $\mathbf{K}$ can have an arbitrary number of components. An analogous equation is satisfied along the η coordinate direction. The factor '2' appears because one must consider all four components of the stress tensor when forming the derivative, even though the off-diagonal terms have equal values.

If the transformed equations are affixed into a matrix form, the governing partial differential momentum and yield surface equations in characteristic coordinates become

$$\mathbf{P}_\sigma \frac{\partial \mathbf{Z}_\sigma}{\partial \xi} + \mathbf{Q}_\sigma \frac{\partial \mathbf{Z}_\sigma}{\partial \eta} + \mathbf{R} = 0. \tag{12}$$

Coefficient matrices $\mathbf{P}_\sigma$ and $\mathbf{Q}_\sigma$ and the forcing vector $\mathbf{R}$ are

$$\mathbf{P}_\sigma = \begin{bmatrix} 2Y_\eta \dfrac{\partial \phi}{\partial \sigma_{xy}} + X_\eta \dfrac{\partial \phi}{\partial \sigma_{xx}} & X_\eta \dfrac{\partial \phi}{\partial \sigma_{yy}} \\ -Y_\eta \dfrac{\partial \phi}{\partial \sigma_{xx}} & -Y_\eta \dfrac{\partial \phi}{\partial \sigma_{yy}} - 2X_\eta \dfrac{\partial \phi}{\partial \sigma_{xy}} \end{bmatrix} \tag{13}$$

$$\mathbf{Q} = \begin{bmatrix} -2Y_\xi \dfrac{\partial \phi}{\partial \sigma_{xy}} - X_\xi \dfrac{\partial \phi}{\partial \sigma_{xx}} & -X_\xi \dfrac{\partial \phi}{\partial \sigma_{yy}} \\ Y_\xi \dfrac{\partial \phi}{\partial \sigma_{xx}} & Y_\xi \dfrac{\partial \phi}{\partial \sigma_{yy}} + 2X_\xi \dfrac{\partial \phi}{\partial \sigma_{xy}} \end{bmatrix} \tag{14}$$

$$\mathbf{R} = \begin{Bmatrix} X_\eta \dfrac{\partial \phi}{\partial \mathbf{K}} : \dfrac{\partial \mathbf{K}}{\partial \xi} - X_\xi \dfrac{\partial \phi}{\partial \mathbf{K}} : \dfrac{\partial \mathbf{K}}{\partial \eta} + 2J \dfrac{\partial \phi}{\partial \sigma_{xy}} \left[\tau_{ax} - \tau_{wx} + mf\left(v - v_g\right)\right] \\ -Y_\eta \dfrac{\partial \phi}{\partial \mathbf{K}} : \dfrac{\partial \mathbf{K}}{\partial \xi} + Y_\xi \dfrac{\partial \phi}{\partial \mathbf{K}} : \dfrac{\partial \mathbf{K}}{\partial \eta} + 2J \dfrac{\partial \phi}{\partial \sigma_{xy}} \left[\tau_{ay} - \tau_{wy} - mf\left(u - u_g\right)\right] \end{Bmatrix}. \tag{15}$$

This set of partial differential equations can be reduced to a set that has a derivative only in the ξ coordinate direction if and only if $\mathbf{Q}_\sigma \partial \mathbf{Z}_\sigma / \partial \eta = 0$. A nontrivial partial derivative solution then exists only if the determinant of the coefficient matrix is zero, $\left|\mathbf{Q}_\sigma\right| = 0$. Expanding the elements of $\mathbf{Q}_\sigma$ defines the stress characteristic along the ξ-direction

$$Y_\xi Y_\xi \frac{\partial \phi}{\partial \sigma_{yy}} + 2X_\xi Y_\xi \frac{\partial \phi}{\partial \sigma_{xy}} + X_\xi X_\xi \frac{\partial \phi}{\partial \sigma_{xx}} = 0 . \tag{16}$$

To this end let the ξ-axis make angle θ_ξ with the x-axis, i.e., $X_\xi = \cos\theta_\xi$ and $Y_\xi = \sin\theta_\xi$ so

$$\tan\theta_\xi = \frac{-\dfrac{\partial \phi}{\partial \sigma_{xy}} \pm \sqrt{\left(\dfrac{\partial \phi}{\partial \sigma_{xy}}\right)^2 - \dfrac{\partial \phi}{\partial \sigma_{yy}} \dfrac{\partial \phi}{\partial \sigma_{xx}}}}{\dfrac{\partial \phi}{\partial \sigma_{yy}}} . \tag{17}$$

Two real solutions θ_ξ exist if the discriminant is positive (hyperbolic), one if zero (parabolic), and none if negative (elliptic). An identical result follows for θ_η, the second characteristic coordinate direction η, when we set $\left|\mathbf{P}_\sigma\right| = 0$.

Finally, the governing equations along the ξ characteristic coordinate satisfy

$$\mathbf{P}_\sigma \frac{\partial \mathbf{Z}_\sigma}{\partial \xi} + \mathbf{R} = 0 . \tag{18}$$

and along the η characteristic coordinate satisfy

$$\mathbf{Q}_\sigma \frac{\partial \mathbf{Z}_\sigma}{\partial \eta} + \mathbf{R} = 0 . \tag{19}$$

As desired, each of these equations is an ordinary differential equation. This fact implies that the solution $\mathbf{Z}_\sigma$ can be discontinuous across the characteristic line. While tractions must be continuous across the characteristic line, the stress component along the line can be discontinuous across it.

Velocity characteristics

The highest order derivatives of velocity appear in the flow rule equations. Following *Pritchard* (1988, 2008), form the ratios of the shear component to each normal component of the flow rule, thus reducing the flow rule to two equations and eliminating the scalar λ. These two equations involve velocity derivatives and stress-dependent derivatives of the potential surface

$$\frac{\partial\psi}{\partial\sigma_{xx}}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)-2\frac{\partial\psi}{\partial\sigma_{xy}}\frac{\partial u}{\partial x}=0$$

$$\frac{\partial\psi}{\partial\sigma_{yy}}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)-2\frac{\partial\psi}{\partial\sigma_{xy}}\frac{\partial v}{\partial y}=0 \quad . \qquad (20)$$

The velocity characteristic analysis follows the stress characteristic analysis by affixing velocity components into a two-component solution vector (again shown as the transpose

$$\mathbf{Z}_v{}^t=\{\,u \quad v\,\}. \qquad \mathbf{(21)}$$

Two coordinates ξ and η are again introduced, where we reuse the coordinate mappings to simplify notation. Along each such direction, the governing equations have a derivative only in that coordinate, not the other. We again affix all transformed equations into a matrix form so that the governing partial differential equation expressed in characteristic coordinates become

$$\mathbf{P}_v\frac{\partial\mathbf{Z}_v}{\partial\xi}+\mathbf{Q}_v\frac{\partial\mathbf{Z}_v}{\partial\eta}=0 \quad . \qquad \mathbf{(22)}$$

The coefficient matrices $\mathbf{P}_v$ and $\mathbf{Q}_v$ are

$$\mathbf{P}_v=\begin{bmatrix} -X_\eta\dfrac{\partial\psi}{\partial\sigma_{xx}}-2Y_\eta\dfrac{\partial\psi}{\partial\sigma_{xy}} & Y_\eta\dfrac{\partial\psi}{\partial\sigma_{xx}} \\ -X_\eta\dfrac{\partial\psi}{\partial\sigma_{yy}} & Y_\eta\dfrac{\partial\psi}{\partial\sigma_{yy}}+2X_\eta\dfrac{\partial\psi}{\partial\sigma_{xy}} \end{bmatrix} \qquad \mathbf{(23)}$$

$$\mathbf{Q}_v=\begin{bmatrix} X_\xi\dfrac{\partial\psi}{\partial\sigma_{xx}}+2Y_\xi\dfrac{\partial\psi}{\partial\sigma_{xy}} & -Y_\xi\dfrac{\partial\psi}{\partial\sigma_{xx}} \\ X_\xi\dfrac{\partial\psi}{\partial\sigma_{yy}} & -Y_\xi\dfrac{\partial\psi}{\partial\sigma_{yy}}-2X_\xi\dfrac{\partial\psi}{\partial\sigma_{xy}} \end{bmatrix}. \qquad \mathbf{(24)}$$

The velocity characteristic equations are derived by analogy with the stress characteristic equations. The ξ velocity characteristic coordinate direction satisfies

$$Y_\xi Y_\xi \frac{\partial \psi}{\partial \sigma_{yy}} + 2X_\xi Y_\xi \frac{\partial \psi}{\partial \sigma_{xy}} + X_\xi X_\xi \frac{\partial \psi}{\partial \sigma_{xx}} = 0 \tag{25}$$

and an analogous relationship for the η velocity characteristic coordinate.

It is illustrative to consider the time dependent motion χ such that $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$, where $\mathbf{x}$ is position at time t of particle at position $\mathbf{X}$ in a reference configuration. Under the deformation gradient $\mathbf{F} = \partial \boldsymbol{\chi} / \partial \mathbf{X}$ a unit vector **N** deforms to a stretched and rotated vector $\mathbf{n} = \mathbf{F} \cdot \mathbf{N}$. Now the time derivative is $\dot{\mathbf{n}} = \dot{\mathbf{F}} \cdot \mathbf{N}$, where $\dot{\mathbf{F}} = \mathbf{L} \cdot \mathbf{F}$ and $\mathbf{L}$ is the velocity gradient $\mathbf{L} = \partial \mathbf{v} / \partial \mathbf{x}$. Given these definitions, we can easily see that $\dot{\mathbf{n}} = \mathbf{L} \cdot \mathbf{n}$. But if the length squared of **n** is $l^2 = \mathbf{n}^t \cdot \mathbf{n}$, its material rate of change is

$$2l\dot{l} = \mathbf{n}^t \cdot \dot{\mathbf{n}} + \dot{\mathbf{n}}^t \cdot \mathbf{n} = \mathbf{n}^t \cdot \left(\mathbf{L} + \mathbf{L}^t\right) \cdot \mathbf{n} \tag{26}$$

and the stretching tensor $\mathbf{D} = \left(\mathbf{L} + \mathbf{L}^t\right)/2$ so that $l\dot{l} = \mathbf{n}^t \cdot \mathbf{D} \cdot \mathbf{n}$. Along the ξ characteristic coordinate we can expand in terms of Cartesian components to find

$$l^{-1}\dot{l} = D_{xx} X_\xi X_\xi + 2D_{xy} X_\xi Y_\xi + D_{yy} Y_\xi Y_\xi \,. \tag{27}$$

But the right side is simply the definition of the ξ velocity characteristic coordinate direction if **D** satisfies the flow rule, which is true whenever the stress state is plastic. Therefore there is zero stretching along a characteristic line ($\dot{l} = 0$) and the component of velocity in the characteristic direction must be constant. A similar result is true along the η velocity characteristic coordinate direction. These basic facts suggest that we express velocity in terms of characteristic coordinates as the primary variables in a numerical solution.

Velocity and Velocity gradient Component Relationships

The characteristic velocity components in a numerical solution must be transformed into Cartesian coordinates. In Cartesian coordinates, we may write velocity $\mathbf{v}$ as the sum of components in the x and y directions

$$\mathbf{v} = v_x \mathbf{e}_x + v_y \mathbf{e}_y \tag{28}$$

where $\mathbf{e}_x$ and $\mathbf{e}_y$ are unit vectors aligned with the x and y coordinate axes. In characteristic coordinates, the velocity can be expressed analogously as

$$\mathbf{v} = v_\xi \mathbf{e}_\xi + v_\eta \mathbf{e}_\eta \tag{29}$$

where the unit length base vectors along the characteristic directions satisfy

$$\begin{aligned} \mathbf{e}_\xi &= X_\xi \mathbf{e}_x + Y_\xi \mathbf{e}_y \\ \mathbf{e}_\eta &= X_\eta \mathbf{e}_x + Y_\eta \mathbf{e}_y \end{aligned} \,. \tag{30}$$

The basis vectors e_ξ and e_η are curvilinear, have unit length, and are generally non-orthogonal. Substitute them into equation (29) and collect into Cartesian components

$$\begin{aligned} v_x &= X_\xi v_\xi + X_\eta v_\eta \\ v_y &= Y_\xi v_\xi + Y_\eta v_\eta \end{aligned} \,. \tag{31}$$

The velocity gradient $\mathbf{L} = \nabla \mathbf{v}$ may be expressed in characteristic coordinates as

$$\mathbf{L} = \frac{\partial\left(v_\xi \mathbf{e}_\xi + v_\eta \mathbf{e}_\eta\right)}{\partial \xi} \otimes \mathbf{e}_\xi + \frac{\partial\left(v_\xi \mathbf{e}_\xi + v_\eta \mathbf{e}_\eta\right)}{\partial \eta} \otimes \mathbf{e}_\eta \qquad \textbf{(32)}$$

The spatial derivatives of the direction cosines can be determined because they are trig functions. The resulting relationship between Cartesian and characteristic velocity components is

$$\begin{aligned}
L_{xx} &= \frac{\partial\left(v_\xi X_\xi + v_\eta X_\eta\right)}{\partial \xi} X_\xi + \frac{\partial\left(v_\xi X_\xi + v_\eta X_\eta\right)}{\partial \eta} X_\eta \\
L_{xy} &= \frac{\partial\left(v_\xi X_\xi + v_\eta X_\eta\right)}{\partial \xi} Y_\xi + \frac{\partial\left(v_\xi X_\xi + v_\eta X_\eta\right)}{\partial \eta} Y_\eta \\
L_{yx} &= \frac{\partial\left(v_\xi Y_\xi + v_\eta Y_\eta\right)}{\partial \xi} X_\xi + \frac{\partial\left(v_\xi Y_\xi + v_\eta Y_\eta\right)}{\partial \eta} X_\eta \\
L_{yy} &= \frac{\partial\left(v_\xi Y_\xi + v_\eta Y_\eta\right)}{\partial \xi} Y_\xi + \frac{\partial\left(v_\xi Y_\xi + v_\eta Y_\eta\right)}{\partial \eta} Y_\eta
\end{aligned} . \qquad \textbf{(33)}$$

These relationships allow the velocity components in the characteristic coordinate basis to be transformed into Cartesian coordinates as input to the constitutive law.

Weak Form of Quasi-Steady Momentum Balance

Numerical schemes such as the finite element method and Galerkin methods satisfy the momentum balance equation in its weak form. Here the equation is multiplied by a set of basis functions that span the solution space and then integrated the product over the problem domain

$$\int_A \varphi\left(mf\mathbf{B}_{\pi/2} + \rho_w C_w \,|\, \mathbf{v} - \mathbf{c} \,|\, \mathbf{B}_w\right) \cdot \left(\mathbf{v} - \mathbf{c}_g\right) da = \int_A \varphi\, \boldsymbol{\tau}_a da + \int_A \varphi \nabla \cdot \boldsymbol{\sigma} da \qquad \textbf{(34)}$$

where φ represents each of the basis functions. The Green-Gauss Theorem converts the stress divergence term into a line integral around the perimeter of A plus an area integral of the gradient of φ times the stress tensor (this is the spatial equivalent of integrating by parts). We find that

$$\int_A \varphi \nabla \cdot \boldsymbol{\sigma}\, da = \oint_L \varphi \boldsymbol{\sigma} \cdot \mathbf{n}\, dl - \int_A \nabla \varphi \cdot \boldsymbol{\sigma}\, da \qquad \textbf{(35)}$$

where L is the perimeter curve around A and $\mathbf{n}$ is the outward unit normal vector. Therefore,

$$\int_A \varphi\left(mf\mathbf{B}_{\pi/2} + \rho_w C_w \left|\mathbf{v} - \mathbf{c}_g\right| \mathbf{B}_w\right) \cdot \left(\mathbf{v} - \mathbf{c}_g\right) da + \int_A \nabla \varphi \cdot \boldsymbol{\sigma}\, da = \int_A \varphi \boldsymbol{\tau}_a\, da + \oint_L \varphi \boldsymbol{\sigma} \cdot \mathbf{n}\, dl \qquad \textbf{(36)}$$

where all internal variables appear on the left hand side of the equation and all external forcing appears on the right hand side. The traction acting on L is $\mathbf{t} = \boldsymbol{\sigma} \cdot \mathbf{n}$. The only spatial gradient is

$$\nabla \varphi = \frac{\partial \varphi}{\partial x} \mathbf{e}_x + \frac{\partial \varphi}{\partial y} \mathbf{e}_y . \qquad (37)$$

This weak form of momentum balance can now be expanded in terms of its Cartesian coordinates or its characteristic components. At present, we believe it will be reasonable to expand the momentum balance equation in Cartesian coordinates. Equation (36) can be solved for the relative ice velocity $\mathbf{v}-\mathbf{c}_g$ when the right hand side is specified.

SPLITTING THE NODES

The central idea of this new solution approach is to express all equations in characteristic coordinates so that discontinuities across the characteristic lines can be accurately and directly described. Thus, we introduce internal boundaries along the characteristic lines through each node. We expect to assume that solutions are continuous between nodes, but reflect the discontinuities across the characteristics at each node. We call this splitting the nodes. As seen in Figure 1, the neighborhood surrounding a node can be divided by the characteristics; there are four areas if the node is hyperbolic, two if parabolic, and one if elliptic. The discontinuities might be treated as sub-scale features. We assume that each part of the node will satisfy a different momentum balance equation and will therefore move somewhat differently.

The weak form of momentum balance is ideal for describing these features. The Green-Gauss Theorem has provided a crucial part of deriving this equation. Within each sub-area the stress tensor is assumed to be continuous and differentiable, across characteristic lines it can be

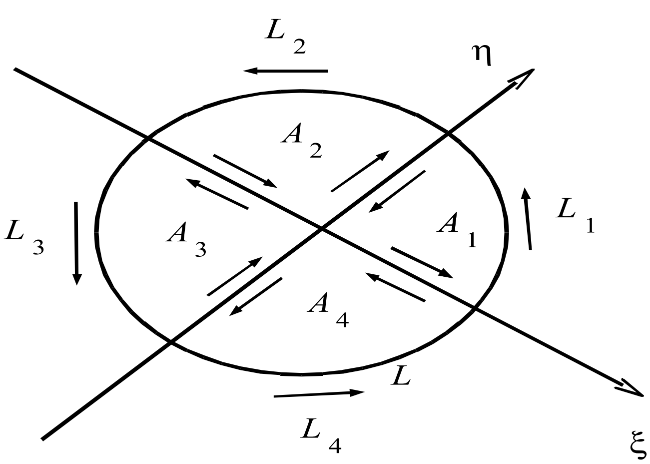

Figure 1. Green-Gauss Theorem and partial areas.

discontinuous. The Green-Gauss Theorem can be applied to the area integral over each partial area. Figure 1 shows a sketch of the hyperbolic case, where the stress tensor can be discontinuous across both the ξ and the η axes. For example, within the partial area A_1, the boundary is L_1, and it is indicated by the separate arrows along each side of the area. The traction along each part of the boundary is $\mathbf{t}=\sigma\cdot\mathbf{n}$.

SUMMARY

This paper describes preliminary ideas for a new ice dynamics numerical model that describes velocity and stress discontinuities explicitly. An anisotropic plasticity model is presented. Behavior can be hyperbolic, parabolic, or elliptic at different interior locations and times with discontinuities possible along characteristic lines within the domain. The characteristic directions are determined as functions of the solution state, and, when advantageous, governing equations are transformed into the characteristic coordinates. The numerical method is expressed in the characteristic coordinates. This approach is generally not used because it is very difficult to introduce the necessarily irregular grid that must be determined as part of the solution. We propose to avoid this difficulty by using a meshless method. Solution nodes will be split into four or two sub-nodes when the solution there is hyperbolic or parabolic, respectively.

A quasi-steady model is required to allow characteristics to be defined. Thus, all inertial effects are neglected and time is a parameter. We focus on determining the solution at a single time. The basic solution variables are the velocity vector and stress tensor. A regular grid of nodes is introduced, but we do not introduce connectivity between the nodes. Thus there are no cells or elements common to most numerical schemes. Given an estimate of the solution, the system type (i.e., hyperbolic, parabolic, or elliptic) can be determined at each node. Nodes will be split at hyperbolic and parabolic points. Velocity can be expressed in characteristic coordinates to account for zero stretching along characteristic lines. Cartesian velocity components can be determined by non-orthogonal coordinate transformation. The anisotropic plasticity constitutive law can be solved for the stress tensor in Cartesian coordinates. Momentum can be balanced at each node, taking into account that discontinuities can exist at a node between the split parts if tractions remain continuous. We assume that the basis for the stress divergence will be limited to the side of the node associated with each sub-node. We expect the traction vector between sub-nodes to be a needed variable. This logic has not yet been developed, but it cannot depend on a fixed connectivity between nodes because changing orientations of the characteristics will make such a fixed choice useless. Leads, rafts, and ridges that form at a node will be considered sub-scale features to be described as part of the icescape. Ice conditions will evolve throughout the problem domain. A numerical code will begin by solving one dimensional problems because these problems require less complex logic.

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