PROBABILISTIC MODELING OF ICE LOADS USING COMPOSITE DISTRIBUTIONS FOR ICE REGIME PARAMETERS

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ABSTRACT

In the paper the problem of ice loads distribution on extreme intervals is considered. Usually finding extreme values exceeding probabilities is connected with considering of ice regime parameters distribution's "tails".

In the paper there are two approaches to describe of ice loads distribution "tail" presented. By the first approach, the probability density functions of ice regime parameters are estimated by piecewise linear functions, and then modeling of those parameters by the method of composition is conducted. After that, design formulas for ice pressure and ice loads easily could be employed, and quantiles of interest could be estimated statistically.

In the second approach only "tails" of ice regime parameters are considered. For the "tails" the rectangular distribution was offered, and the theorem about the distribution of ice loads calculated by design formula was proofed. That theorem was used to calculate quantiles of ice loads. The example, which demonstrated good coincidence of both methods, was considered.

INTRODUCTION

The problem of ice loads probabilistic modeling needs appropriate tools for its solution. One of approaches to this problem, using stochastic processes model, was already offered by the authors earlier (Zvyagin and Sazonov, 2013).

In the last 50 years (Korzhavin 1962; Loset et al. 1999) a number of formulas which relate ice regime parameters with ice pressure or ice loads were offered. These formulas could be applied for building stochastic models where maximum annual ice pressure is considered as random variable. In fact, the event "maximal ice pressure next year will exceed the specific level" is random, and thus the probability of this event can be related with maximal ice pressure models, given by investigators as well as in national codes. Therefore, the problem of estimating the probability of mentioned event reduces to the problem of random ice loads quantile finding.

In the specific conditions ice regime parameters, considered as random, can be described by different distribution laws. With respect of this,iIn the paper the problem of random maximum annual ice load quantiles estimating is solved in two different ways.

METHOD OF COMPOSITION

Let us consider histogram of ice strength or ice thickness data measured in the local Arctic area of interest. Let us fix the number m of intervals, on which the entire data range was split to plot this histogram. Then we can build the next piecewise linear function, based on the relative interval frequencies ν_i, $i = 1..m$, to estimate unknown theoretical probability density function (PDF) of the random data:

$$\tilde{f}(x)=\begin{cases}\nu_1, & x_0 \le x \le x_1;\\ \nu_2, & x_1 < x \le x_2;\\ \dots & \dots\\ \nu_m, & x_{m-1} < x \le x_m.\end{cases} \quad (1)$$

In fact, the piecewise linear estimator $\tilde{f}(x)$ provides best fit to corresponding theoretical PDF in the sense of Chi-squared test.

The idea of composition method is to use simple probability density functions to compose of them more complicated PDF. Particularly, we shall represent (1) by the next expression:

$$\tilde{f}(x)=\sum_{i=1}^{m}\nu_i f_i, \qquad f_i=\begin{cases}0, & x \le x_{i-1}\\ \dfrac{1}{x_i - x_{i-1}} & x_{i-1} < x \le x_i\\ 0, & x > x_i\end{cases} \quad (2)$$

Here f_i are PDFs of uniformly distributed random variables, which are easy to simulate by any appropriate statistical software.

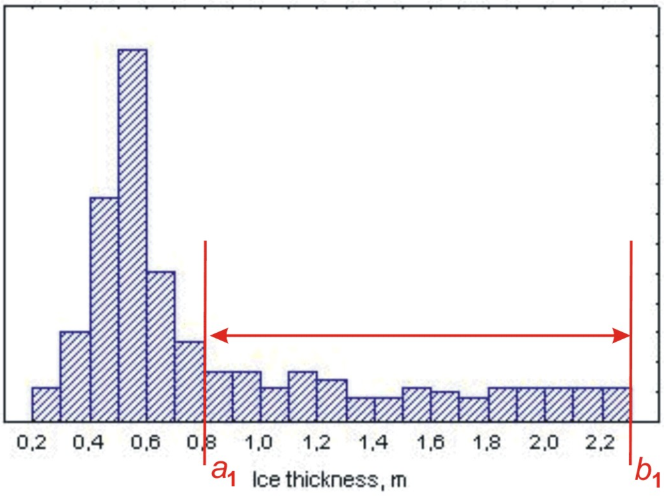

Figure 1. Ice thickness density histogram, presented in the paper (Pfaffling et al. 2006).

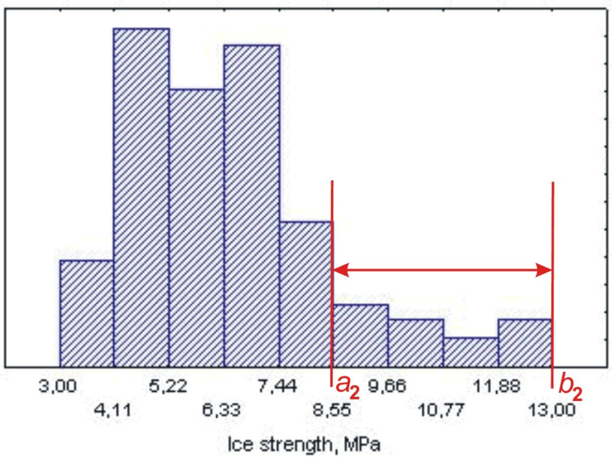

Figure 2. Ice strength density histogram, given for Spitzbergen area by (Sinitsyna et. al. 2013), based on field measurements on 22 500 m^2, presented in (Shafrova and Moslet, 2006).

Nowadays in the papers dedicated to ice regime parameters there are no univocal opinion about distribution laws of ice strength or ice thickness. Usually the standard distribution law, which has a good fit to the empirical histogram, has also a "tail" on the right side, which continues up to the infinity. This model provides non-zero probabilities for practically impossible events. At the same time, the right parts of a number of histograms for ice regime parameters (Sinitsyna et al. 2013, Pfaffling et al. 2006, Masaki et. al. 2006) have right "tails" which could be described by rectangular distribution. We can see that feature for histograms presented in the Figures 1 and 2.

The ISO 19906 code (2010) recommends estimating maximum global pressure P on structure according to the next formula:

$$P = \tilde{c} \cdot h^{\alpha} \cdot R \tag{3}$$

where h is the ice thickness, R is the compressive ice strength, and $\tilde{c}$ is some constant value, which depends on the structure width. Loset et al. (1999) in their overview compared a number of design formulas, which relate ice thickness, ice strength and ice pressure or loads in the way of (3).

If ice condition parameters h and R are random variables, then global pressure P is also random variable, and distribution law of P will depend on distribution laws of h and R. According to ISO code (2010) and book of Palmer and Croasdale (2013), $\tilde{c} = d^{-0,16}$, and the power value α is negative. In Palmer and Croasdale (2013) the next range for this parameter in dependence of ice thickness is recommended: $-0.5 \le \alpha \le -0.3$. In this paper, having ice thickness h random we shall take α as nonrandom value, and particularly $\alpha = -0,3$.

Thus, for maximum effective global pressure we have according (Palmer and Croasdale 2013):

$$P = d^{-0,16} h^{-0,3} R$$

To estimate global force on the construction, we shall multiply the global pressure by the contact area, calculated in the simplest way: as a product of structure length and ice thickness. Thus, from (3) for global ice force we'll have:

$$F = ch^{0,7} R = d^{0,84} h^{0,7} R, \tag{4}$$

where we assume that c is constant. Let us consider the structure of width $d = 2$ m, and therefore we'll have $c = d \cdot d^{-0,16} = d^{0,84} \approx 1,8$. Having distributions for ice thickness and ice strength we can say, that most adverse situation, which can be considered, is when these characteristics are at their maximum values: $F_{\max} = d^{0,84} 2,3^{0,7} 13 \approx 42$ MN. But what is probability of such event?

Let assume that h and R are independent random variables. We can use the method of composition to simulate h and R according to histograms presented in Figures 1 and 2, and get the histogram of global force F applying relation (4). This histogram presented in the Figure 3. Estimators of upper quantiles are presented in the Table 1.

Table 1. Point estimators of global force p-quantiles (simulation results).

p	β	Quantile estimator	p	β	Quantile estimator
0,5	0,5	8,6	0,975	0,025	25,75
0,9	0,1	18,13	0,99	0,01	31,62
0,95	0,05	21,97	0,995	0,005	35,29

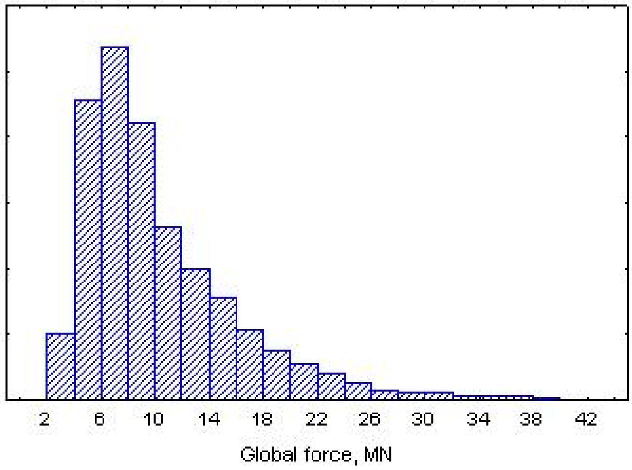

Figure 3. Histogram of simulated global ice force on construction of 2 m width.

DISTRIBUTION OF ICE LOADS ON EXTREME INTERVALS

Let us focus on high values of ice thickness and ice strength and take "tails" of h and R probability density functions as rectangular, and all other parts of these PDFs leave undefined:

$$\tilde{f}(x)=\begin{cases}\tilde{f}_1(x) & x_0 \le x < a;\\ \dfrac{p_1}{b-a} & a \le x \le b.\end{cases}$$

Here $b = x_m$, $a \le x_{m-1}$ and $\int_{x_0}^{a} \tilde{f}_1(x)dx = 1 - p_1$.

Let us denote with $\tilde{f}_h(x)$ the PDF of random ice thickness, and with $\tilde{f}_R(x)$ − the PDF of random compressive strength:

$$\tilde{f}_h(x)=\begin{cases}\tilde{f}_{1h}(x) & x_{h0} \le x < a_1;\\ \dfrac{p_1}{b_1-a_1} & a_1 \le x \le b_1.\end{cases} \qquad \tilde{f}_R(x)=\begin{cases}\tilde{f}_{1R}(x) & x_{R0} \le x < a_2;\\ \dfrac{p_2}{b_2-a_2} & a_2 \le x \le b_2.\end{cases} \qquad (5)$$

Then the probability of random event $A = "h \ge a_1" \wedge "R \ge a_2"$ is $p_1 p_2$.

For ice regime parameters histograms presented in the Figures 1 and 2 we can provide the next PDF estimators, taken respectively from papers (Pfaffling et al. 2006) and (Sinitsyna et. al. 2013):

$$\tilde{f}_h(x)=\begin{cases}\tilde{f}_{1h}(x) & 0{,}2 \le x < 0{,}8;\\ \dfrac{p_1}{1{,}5} & 0{,}8 \le x \le 2{,}3.\end{cases} \qquad \tilde{f}_R(x)=\begin{cases}\tilde{f}_{1R}(x) & 3{,}0 \le x < 8{,}55;\\ \dfrac{p_2}{4{,}45} & 8{,}55 \le x \le 13.\end{cases}$$

Here $p_1 = 0{,}359$, $p_2 = 0{,}142$. For the further reasoning about maximal values and their probabilities, we shall take only "right part" of both distributions. Thus, we shall consider scenario, when strength and thickness are both from intervals of maximal values:

$$g_h(x)=\begin{cases}0, & x < 0{,}8;\\ \dfrac{2}{3} & 0{,}8 \le x < 2{,}3;\\ 0, & x > 2{,}3.\end{cases} \qquad (6a) \qquad g_R(x)=\begin{cases}0, & x < 8{,}55;\\ \dfrac{1}{4{,}45} & 8{,}55 \le x < 13{,}0;\\ 0, & x > 13{,}0.\end{cases} \qquad (6b)$$

We shall denote this scenario as "Scenario of maximal values". The probability of it is $p_1 p_2 \approx 0{,}051$. Let us define the distribution of global ice force F_1 on the structure in the case of this scenario and with respect of relation (4).

We shall take into account that all of bounds a_1^α, b_1^α, a_2, b_2 (5) are greater than zero because of physical sense of considered random variables, and $a_1^\alpha \neq b_1^\alpha$, $a_2 \neq b_2$. At first let us provide a helpful corollary.

Corollary. If random variable ξ is uniformly distributed on the segment $[a,b]$, $a > 0$, $b > 0$, then random variable ξ^α, $\alpha = const$, $\alpha \neq 0$, $\alpha \neq 1$, has the next probability density function:

if $\alpha > 0$,

$$f_\xi(x) = \begin{cases} 0, & x < a^\alpha; \\ \dfrac{x^{\frac{1}{\alpha}-1}}{\alpha(b-a)}, & a^\alpha \leq x \leq b^\alpha; \\ 0, & x > b^\alpha; \end{cases} \qquad (7a)$$

if $\alpha < 0$,

$$f_\xi(x) = \begin{cases} 0, & x < b^\alpha; \\ -\dfrac{x^{\frac{1}{\alpha}-1}}{\alpha(b-a)}, & b^\alpha \leq x \leq a^\alpha; \\ 0, & x > a^\alpha. \end{cases} \qquad (7b)$$

The PDF curve of random variable h^α with $\alpha = 0.7$ built according (7), where h has PDF (6 a), is presented in the Figure 4.

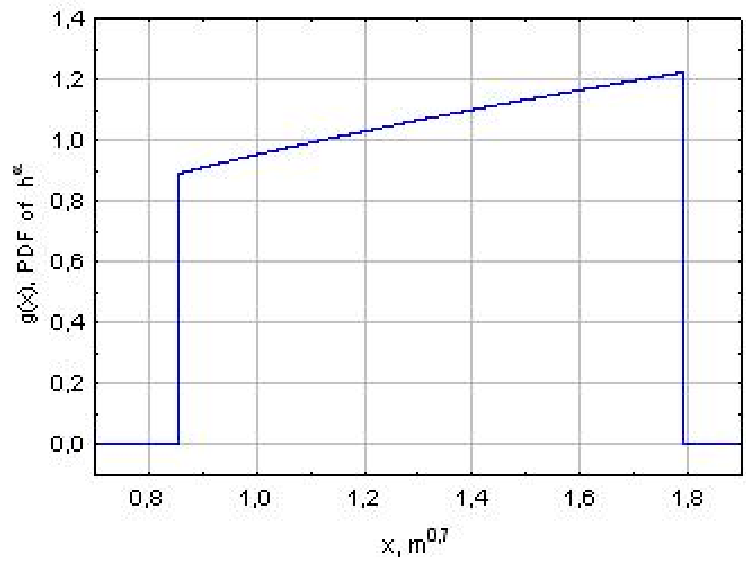

Figure 4. Curve of PDF of random variable h^α, according to 7 a).

Theorem. If random variable ξ is uniformly distributed on the segment $[a_1, b_1]$, and random variable η is uniformly distributed on the segment $[a_2, b_2]$, and $a_1 > 0$, $b_1 > 0$, $a_2 > 0$, $b_2 > 0$, $\alpha = const$, $\alpha > 0$, $\alpha \neq 1$, then random variable $X = c \cdot \xi^\alpha \cdot \eta$ has the next probability density function, if $a_1^\alpha b_2 < a_2 b_1^\alpha$:

$$f^*(x)=\begin{cases}\dfrac{1}{c(1-\alpha)(b_2-a_2)(b_1-a_1)}\left[\left(\dfrac{x}{ca_2}\right)^{\frac{1}{\alpha}-1}-a_1^{1-\alpha}\right], & ca_1^{\alpha}a_2\le x\le ca_1^{\alpha}b_2;\\ \dfrac{x^{\frac{1}{\alpha}-1}c^{-\frac{1}{\alpha}}}{(1-\alpha)(b_2-a_2)(b_1-a_1)}\left[b_2^{1-\frac{1}{\alpha}}-a_2^{1-\frac{1}{\alpha}}\right], & ca_1^{\alpha}b_2<x<ca_2b_1^{\alpha};\\ \dfrac{1}{c(1-\alpha)(b_2-a_2)(b_1-a_1)}\left[b_1^{1-\alpha}-\left(\dfrac{x}{cb_2}\right)^{\frac{1}{\alpha}-1}\right], & ca_2b_1^{\alpha}\le x\le cb_1^{\alpha}b_2;\end{cases}\qquad (8\text{ a})$$

And the next probability density function, if $a_2b_1^{\alpha}<a_1^{\alpha}b_2$:

$$f^*(x)=\begin{cases}\dfrac{1}{c(1-\alpha)(b_2-a_2)(b_1-a_1)}\left[\left(\dfrac{x}{ca_2}\right)^{\frac{1}{\alpha}-1}-a_1^{1-\alpha}\right], & ca_1^{\alpha}a_2\le x\le ca_2b_1^{\alpha};\\ \dfrac{1}{c(1-\alpha)(b_2-a_2)(b_1-a_1)}\left[b_1^{1-\alpha}-a_1^{1-\alpha}\right], & ca_2b_1^{\alpha}<x<ca_1^{\alpha}b_2;\\ \dfrac{1}{c(1-\alpha)(b_2-a_2)(b_1-a_1)}\left[b_1^{1-\alpha}-\left(\dfrac{x}{cb_2}\right)^{\frac{1}{\alpha}-1}\right], & ca_1^{\alpha}b_2\le x\le cb_1^{\alpha}b_2.\end{cases}\qquad (8\text{ b})$$

PROOF OF THE THEOREM 1

Let us consider $a_1^{\alpha}b_2<a_2b_1^{\alpha}$. Then for $ca_1^{\alpha}a_2\le x\le ca_{1_1}^{\alpha}b_2$ we have:

$$f^*(x)=\left(\frac{1}{c\alpha(b_2-a_2)(b_1-a_1)}\int_{a_1^{\alpha}}^{\frac{x}{ca_2}}\left(\frac{x}{y}-ca_2\right)y^{\frac{1}{\alpha}-1}dy\right)'=\frac{1}{c(1-\alpha)(b_2-a_2)(b_1-a_1)}\left[\left(\frac{x}{ca_2}\right)^{\frac{1}{\alpha}-1}-a_1^{1-\alpha}\right]$$

For $ca_1^{\alpha}b_2<x<ca_2b_1^{\alpha}$ we have:

$$f^*(x)=\left(Prob\left(\xi^{\alpha}\eta<a_1^{\alpha}cb_2\right)+\frac{1}{c\alpha(b_2-a_2)(b_1-a_1)}\int_{ca_2}^{cb_2}dz\int_{\frac{a_1^{\alpha}cb_2}{z}}^{\frac{x}{z}}y^{\frac{1}{\alpha}-1}dy\right)'=\frac{x^{\frac{1}{\alpha}-1}c^{-\frac{1}{\alpha}}}{(1-\alpha)(b_2-a_2)(b_1-a_1)}\left[b_2^{1-\frac{1}{\alpha}}-a_2^{1-\frac{1}{\alpha}}\right]$$

For $ca_2b_1^{\alpha}\le x\le cb_1^{\alpha}b_2$ we have:

$$f^*(x)=\left(1-\frac{1}{c\alpha(b_2-a_2)(b_1-a_1)}\int_{\frac{x}{cb_2}}^{b_1^{\alpha}}y^{\frac{1}{\alpha}-1}\left(cb_2-\frac{x}{y}\right)dy\right)'=\frac{1}{c(1-\alpha)(b_2-a_2)(b_1-a_1)}\left[b_1^{1-\alpha}-\left(\frac{x}{cb_2}\right)^{\frac{1}{\alpha}-1}\right].$$

The proof for the second part of the theorem when $a_2b_1^{\alpha}<a_1^{\alpha}b_2$ is similar to proof provided above.

EXAMPLE OF APPLICATION

Then we can consider $f^*(x)$ (8 a-b) as probability density function of random ice force F^* in "scenario of maximum values". The curve of PDF $f^*(x)$ in the case of $a_1 = 0{,}8$, $b_1 = 2{,}3$, $a_2 = 8{,}55$, $b_2 = 13{,}0$, $\alpha = 0{,}7$, $c = 1{,}8$ is presented in the Figure 5.

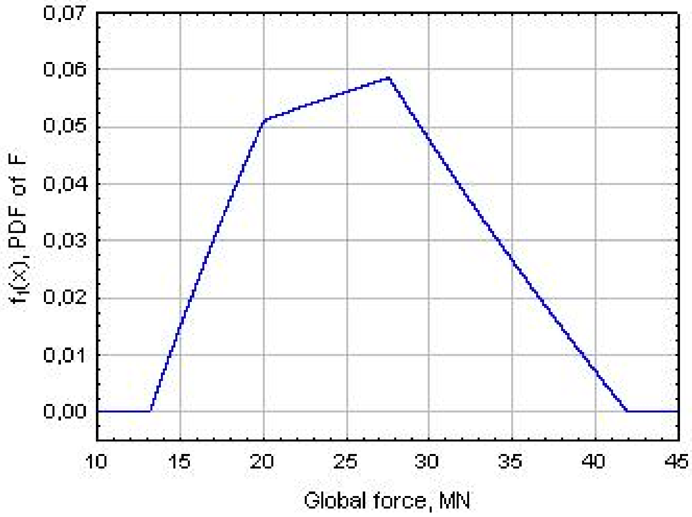

Figure 5. PDF of random variable $F^* = 1{,}8 \cdot h^{0{,}7} \cdot R$ where h and R are random variables with PDF's (6 a) and (6 b) respectively.

The $\tilde{p}$-quantiles of random variable F^*, which PDF is plotted in the Figure 5, with values of $\tilde{\beta} = 1 - \tilde{p}$ are presented in the Table 2.

Table 2. Quantiles of global load in the case of "Scenario of maximal values".

$\tilde{p}$	$\tilde{\beta}$	$\tilde{p}$-quantile	$\tilde{p}$	$\tilde{\beta}$	$\tilde{p}$-quantile
0,5	0,5	25,9	0,975	0,025	38,25
0,9	0,1	34,64	0,99	0,01	39,59
0,95	0,05	36,74	0,995	0,005	40,27

As soon as $\alpha > 0$, we can say that event "Global ice force will be greater than $\max(ca_1^\alpha b_2, cb_1^\alpha a_2)$", where parameter c is taken according to formula (4), will occur only when $h \in [a_1, b_1]$ and at the same time $R \in [a_2, b_2]$. It means that $(1-\tilde{\beta})$-quantiles of F^* will be $(1 - p_1 p_2 \tilde{\beta})$-quantiles of F, for $F > \max(ca_1^\alpha b_2, cb_1^\alpha a_2)$.

In our case $\max(ca_1^\alpha b_2, cb_1^\alpha a_2) = 27{,}57$.

In the Table 3 we can compare estimators of p-quantiles, got by probabilistic modeling according to the method of composition, and corresponding values of $\tilde{p}$-quantiles calculated for theoretical distribution evaluated in Theorem 1. The coincidence is very good in respect of inaccuracy of statistical estimation of p-quantiles of global force F.

Thus, both of these methods could be used to estimate p-quantiles of global force F on the construction, if the model for F is of type (4), and experimental histograms for ice thickness and ice strength are known. The influence of contact area could be taken into account in calculations by usage of appropriate constant coefficients, related to construction width.

Table 3. Comparison of estimators of F quantiles got by modeling, and corresponding F^* quantiles got according to formulas (8).

p	β=1-p	p-quantile of F, got by modeling (MPa)	$\tilde{\beta} = 1 - \tilde{p}$	$\tilde{p}$	$\tilde{p}$-quantile of F^* (MPa)
0,98	0,02	27,13	0,392	0,608	27,76
0,985	0,015	29,42	0,294	0,706	29,6
0,99	0,01	31,62	0,196	0,804	31,8
0,995	0,005	35,29	0,098	0,902	34,71

OUTCOMES

In the paper proposed two methods of quantiles estimation. These methods are based on the formulas for ice pressure and ice loads which are used, with some variations, by a number of investigators (Korzhavin 1962, Loset et al. 1999) as well as in ISO standard (2010). Also method needs the presence of experimental histograms for ice regime parameters.

The first method employs the probabilistic modeling of ice pressure/loads using method of composition. Thus we can get a statistics for parameter of interest, and then find point estimators for quantiles.

The second method allows analytical calculating of probabilities of ice loads critical value exceeding, if only this value is relatively close to maximal possible ice load. The method uses probability density function, evaluated in the Theorem 1 for extreme values of ice loads.

The example, provided in the paper, demonstrates very good coincidence for 0,98-0,995 quantiles, found by both of the methods. But these bounds of good coincidence are greatly depend on parameters of ice loads formula and ice regime histograms.

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