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DIMENSIONLESS STRENGTH PARAMETERS FOR FLOATING ICE SHEETS

R. Reid Parmerter
Associate Professor
Dept. of Aeronautics and Astronautics

AIDJEX
(Arctic Ice Dynamics Joint Experiment)
University of Washington

Seattle, Washington
U.S.A.

ABSTRACT

The problem of vertical loadings on floating ice sheets is discussed from the standpoint of dimensional analysis. Important dimensionless parameters are identified, and strength parameters are considered in detail. Two example problems are studied. One involves the question of the height to which ice will pile against a sea wall. This problem is directly related to the question of the maximum height of pressure ridges. Rafting serves as a second example. The maximum thickness of ice which will raft is found to depend on the square of the strength.

The results show how scale experiments can be designed to yield data applicable to full scale problems. The dimensional parameters also provide an efficient means of presenting either analytical or experimental data.

BASIC EQUATIONS AND DIMENSIONAL ANALYSIS

The arctic ice is often subjected to vertical loadings. In nature, these arise when differential accumulation or erosion of ice results in local areas which are out of isostatic equilibrium. When man ventures upon the ice, he often brings heavy equipment which the ice must support. These vertical forces bend the ice sheet, and induce bending stress components σ_{xx}, σ_{xy}, σ_{yy} and transverse shear components τ_{xz} and τ_{yz}, where the x,y coordinates are in the plane of the sheet and z is perpendicular to it. The vertical component σ_{zz} is negligible, except under concentrated loads, where the assumptions of plate theory breakdown.

Structurally, the ice may be approximated by an elastic plate of thickness t on a Winkler [1] foundation with modulus $k = \rho_w g$, where ρ_w is the density of water and g the acceleration of gravity. This approximation is valid provided the plate is not lifted from the water, or totally submerged. The vertical deflection W is governed by the equation (see reference [2])

$$D \nabla^4 W + kW = q(x,y)$$

where the bending stiffness $D = Et^3/12(1 - \nu^2)$ incorporates two mechanical properties of the ice, Youngs modulus E and Poissons ratio ν. The vertical load per unit area is $q(x,y)$. Introducing dimensionless parameters

$$\lambda = \sqrt[4]{k/4D} \qquad \Delta^4 = \frac{\partial^4}{\partial\zeta^4} + 2\,\frac{\partial^4}{\partial\zeta^2\partial\eta^2} + \frac{\partial^4}{\partial\eta^4} \qquad \zeta = \lambda x \quad \eta = \lambda y \quad \omega = W/t$$

this equation can be put in the form

$$\Delta^4\omega + 4\omega = \frac{q_0}{kt} f(\zeta,\eta) \tag{1}$$

The parameter q_0 is a measure of load intensity, while kt is the uniform load required to displace the plate 1/4 thickness, in the idealized situation where the elastic foundation is linear for displacements of this magnitude. In the floating ice sheet case, the foundation ceases to act when the plate is fully submerged or lifted free of the surface. However, the ratio q_0/kt still has a clear physical interpretation. The function $f(\zeta,\eta)$ is a geometrically scaled load distribution function. Let $\omega_0(\zeta,\eta)$ be the solution of equation (1) for a load intensity $q_0 = kt$. Then the function $\omega_0(\zeta,\eta)$ depends only on the function $f(\zeta,\eta)$. Equation (1) is linear: Thus the response ω to any other load intensity q_0, with the same distribution f, is just $\omega = q_0\omega_0/kt$.

The moments in the ice are related to the deformation by

$$M_{ij} = D\left\{ (1 - \nu)\frac{\partial^2 W}{\partial x_i \partial x_j} + \nu\delta_{ij}\frac{\partial^2 W}{\partial x_s \partial x_s} \right\}$$

and the bending stresses at the surface are

$$\sigma_{ij} = \pm\, 6M_{ij}/t^2$$

From these relations, it is clear that the three tensors M_{ij}, $\partial^2 W/\partial x_i \partial x_j$ and σ_{ij} are symmetric, and have the same principal directions. Let n and m denote the directions of maximum and minimum principal stress. Then the maximum normal stress is

$$\sigma_{max} = \sigma_{nn} = \frac{6D}{t^2}\left\{ \frac{\partial^2 W}{\partial n^2} + \nu\,\frac{\partial^2 W}{\partial m^2} \right\} \tag{2}$$

Introducing dimensionless parameters

$$\sigma_{max} = \frac{q_0}{\lambda^2 t^2} F_1(\omega_0,\nu) \tag{3}$$

A similar expression applies for the maximum inplane shear* $(\sigma_{nn} - \sigma_{mm})/2$.

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* If σ_{nn} and σ_{mm} have the same sign, the <u>maximum</u> shear is $(\sigma_{nn}-\sigma_{zz})/2 \simeq \sigma_{nn}/2$. This is not an inplane shear, but occurs on the plane making a 45° angle with the directions n and z.

Let σ_c denote a failure level for inplane stress (either normal stress or shear stress). Then, in view of equation (3), a dimensionless failure parameter can be defined

$$\frac{\sigma_c \lambda^2 t^2}{q_0} = F_2(f, \nu) \tag{4}$$

The transverse shear τ in the plate is related to third derivatives of W. In a similar manner, the maximum transverse shear can be put in the form

$$\tau_{max} = \frac{q_0}{\lambda t} F_3(\omega_0, \nu) \tag{5}$$

If τ_c is a failure level in transverse shear then

$$\frac{\tau_c \lambda t}{q_0} = F_4(f, \nu) \tag{6}$$

Equations (4) and (6) are useful for designing experiments on scale models. Full scale data can be obtained by designing a small scale experiment to give the desired values of dimensionless groupings. The groupings also provide a minimum number of variables for the problem, simplifying the graphical presentation of analytical or experimental data. For problems where the distribution function depends only on one dimension, $f = f(\zeta)$, and the boundary conditions are independent of η, the solution W is also independent of η. In this case σ_{max} does not depend on ν, see equation (2), except thru the parameter λ. Similar conclusions apply to τ. Thus the functionals F_1, F_2, F_3, F_4 depend only on the distribution function f.

In simple beam theory, the ratio of bending stress to transverse shear depends on the ratio of beam length L to thickness t, with transverse shear becoming negligible for large L/t. A similar result can be deduced from equation (3) and (5), with the distance $1/\lambda$ playing the role of L. For fixed f, ν, the functionals F_1 and F_3 are constant, and

$$\frac{\tau_{max}}{\sigma_{max}} \approx \lambda t \approx t^{\frac{1}{4}}$$

Thus, bending stress dominates as the sheet becomes thin.*

As an example, consider the case of a concentrated line load acting on an infinite sheet. This one dimensional problem is equivalent to a beam on an elastic foundation, and the beam solution, from [1] yields

$$\frac{\tau_{max}}{\sigma_{max}} = \frac{1}{2} \lambda t$$

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* Plate theory assumes that the deformations due to transverse shear are negligible. If the transverse shear becomes important, plate theory must be modified.

so bending dominates if $t < 2/\lambda$. For typical ice properties, $\lambda t = 0.05$ for $t = 20$ *cm*, and $\lambda t = 0.1$ for $t = 4m$. Thus bending dominates in most problems, except in regions where loads are concentrated over distances of the order of a thickness or less. In these regions, the basic assumptions of plate theory breakdown, and more complex, three dimensional analysis may be required locally.

THE SEA WALL PROBLEM

To illustrate the simplified graphical presentation made possible by equation (4), consider a semi-infinite ice sheet which is forced against a wall (Figure 1).

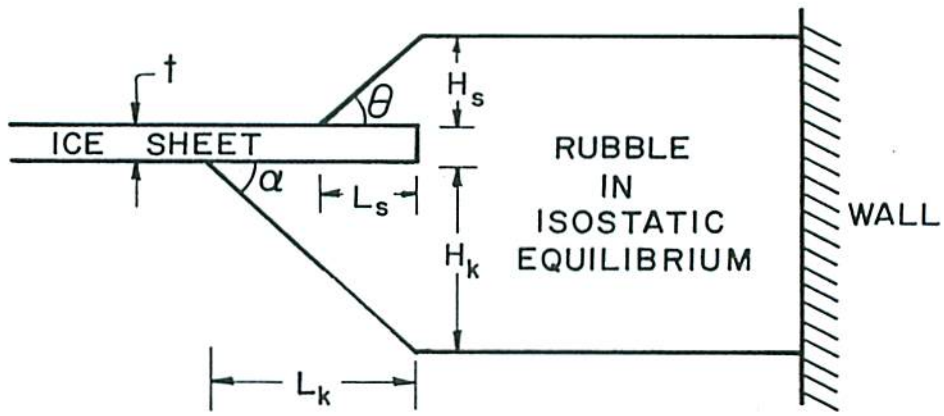

Figure 1

Pieces are broken from the sheet in bending, creating a rubble pile. Rubble tumbles from the pile, forming triangular load distributions on the top and bottom of the sheet, at angles of repose θ and α. The rubble is composed of ice blocks and voids with average density ρ_r. If voids in the rubble are filled with air and water, above and below the surface respectively, then the main rubble pile floats so that

$$H_k = \rho_i H_s / (\rho_w - \rho_i)$$

Thus, the load is completely described by H_s, L_s, L_k, ρ_r, ρ_i and ρ_w.

Once a pile has formed, breaking of the sheet can occur near the point where the sheet enters the pile, due to the unequal triangular loads. It is natural to ask what height H_s is required to load the sheet to breaking level. This problem has been discussed in [3] and [4], where the height H_s is interpreted as a limiting height for pressure ridges. The solution may also be applied to the present problem where ice is forced against a vertical wall. In this case, H_s is the maximum height of rubble which will be piled against the wall. Once this maximum height is reached, the rubble pile will grow laterally rather than vertically. The forces required to push the ice sheet into the rubble pile are also calculated in the references, and provide an estimate of the maximum loads which ice can impose on sea walls.

In dimensionless form, the triangular load distribution is completely specified by

$$\lambda L_s = \frac{\lambda H_s}{tan\theta} \text{ and } \frac{Lk}{L_s} = \frac{\rho_i}{\rho_w - \rho_i}\frac{tan\theta}{tan\alpha} \equiv G$$

while the load intensity may be specified by $q_0 = \rho_r g H_s$. Assume the loading is one dimensional, so ν does not enter F_2. Equation (4) becomes

$$\frac{\sigma_c \lambda^2 t^2}{\rho_r g H_s} = f_1 \left(\frac{\lambda H_s}{tan\theta}, G\right)$$

or equivalently, since we are interested in determining H_s

$$\frac{\lambda H_s}{tan\theta} = f_2 \left(\frac{\sigma_c \lambda^3 t^2}{\rho_r g \ tan\theta}, G\right)$$

Thus H_s, which depends on all the independent variables of the problem

$$H_s = f_3(\sigma_c, E, \nu, \rho_r, \rho_i, \rho_w, t, \theta, \alpha)$$

may conveniently be plotted in dimensionless form as a dimensionless height $\lambda H_s/tan\theta$ vs a dimensionless strength $\sigma_c\lambda^3 t^2/\rho_r g tan\theta$, with geometry G as a parameter. The results are shown in Figure 2.

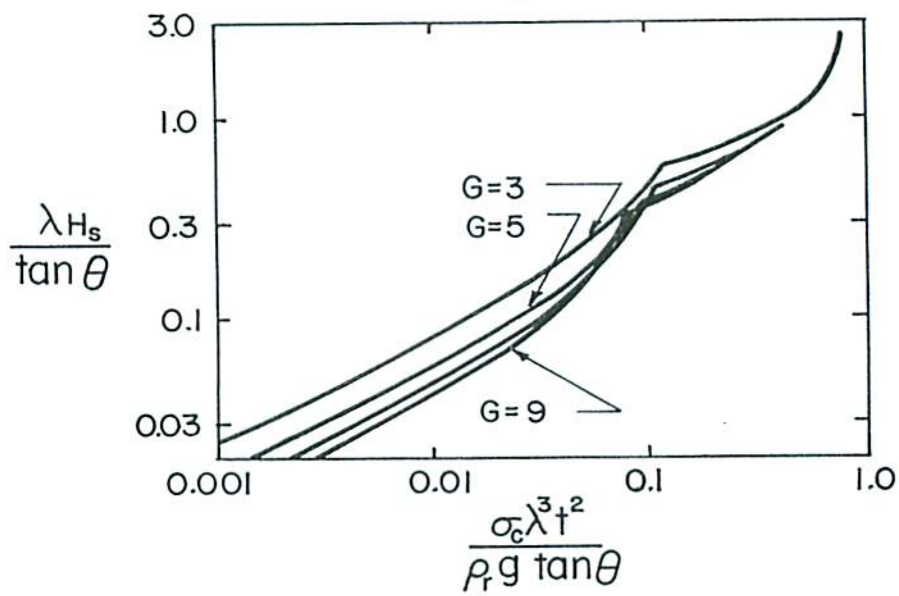

Figure 2

INPLANE FORCES

The problem of a plate on an elastic foundation may be generalized by including the effect of inplane forces. In arctic ice, significant tension forces cannot develop, due to the widespread presence of cracks which do not support tension. However, compressive forces of considerable magnitude are present, as evidenced by compressive processes such as ridging and rafting.

Inplane compressive forces interacting through the vertical deflection of the plate, create moments which tend to increase the displacement. At some critical value of inplane force, buckling can occur, i.e., vertical deflections appear even when vertical loads are not present. Sign conventions for the inplane force per unit length, F_x, F_y, F_{xy}, F_{yx} are shown in Figure 3.

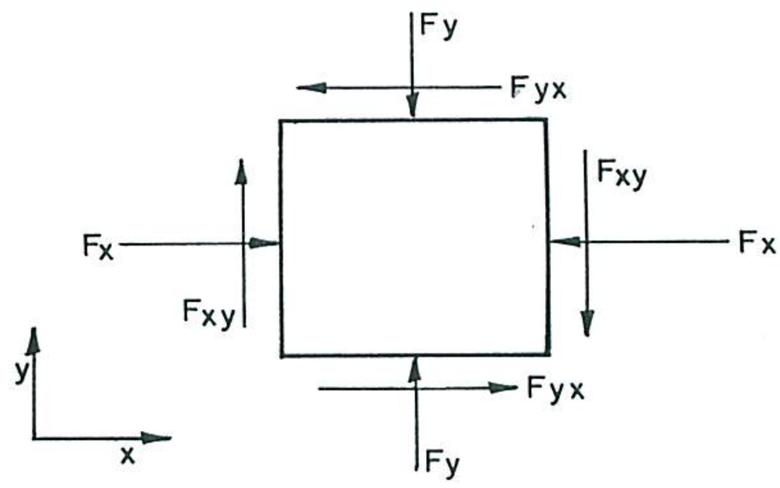

Figure 3

From moment equilibrium, $F_{xy} = F_{yx}$. The differential equation of the plate becomes, see [2]

$$D \nabla^4 W + F_x \frac{\partial^2 W}{\partial x^2} + F_y \frac{\partial^2 W}{\partial y^2} + 2 F_{xy} \frac{\partial^2 W}{\partial x \partial y} + kW = q$$

Introducing dimensionless parameters, the equation takes the form

$$\Delta^4 \omega + 4 \left\{ N_\zeta \frac{\partial^2 \omega}{\partial \zeta^2} + N_\eta \frac{\partial^2 \omega}{\partial \eta^2} + 2 N_{\zeta\eta} \frac{\partial^2 \omega}{\partial \zeta \partial \eta} + \omega \right\} = \frac{q_0}{kt} f(\zeta,\eta) \qquad (7)$$

where

$$N_\zeta = \frac{F_x \lambda^2}{k} \qquad N_\eta = \frac{F_y \lambda^2}{k} \qquad N_{\zeta\eta} = \frac{F_{xy} \lambda^2}{k}$$

are dimensionless inplane force parameters. For fixed inplane forces, equation (7) is linear in ω, so that the principle of superposition may be used. Introducing a function ω_0, as in the first section, we see that in this case, ω_0 depends on the load distribution function f, and on the inplane components. Thus,

$$W = \frac{q_0}{k} F_5(f, N_\zeta, N_\eta, N_{\zeta\eta}) \qquad (8)$$

The dimensionless strength parameter for bending takes the form

$$\frac{\sigma_c \lambda^2 t^2}{q_0} = F_6(f, \nu, N_\zeta, N_\eta, N_{\zeta\eta})$$

For problems where $f = f(\zeta)$, with boundary conditions independent of η, and with the inplane force in the direction of ζ, this relation takes the simple form

$$\frac{\sigma_c \lambda^2 t^2}{q_0} = F_7(f, N_\zeta) \tag{9}$$

RAFTING

The rafting of sea ice is an interesting problem which illustrates the use of equation (9). When two ice sheets are forced together, one sheet sometimes overrides the other, without breaking. The phenomena is usually observed in thin young ice. The inplane force required to cause rafting, and the stresses which are developed in the ice, are considered in detail in [5]. A simplified model has also been studied by Matsuoka [6]. The rafting model assumed in [5] is shown in Figure 4. The two sheets meet along a straight line of sufficient length that the problem may be considered as one-dimensional.

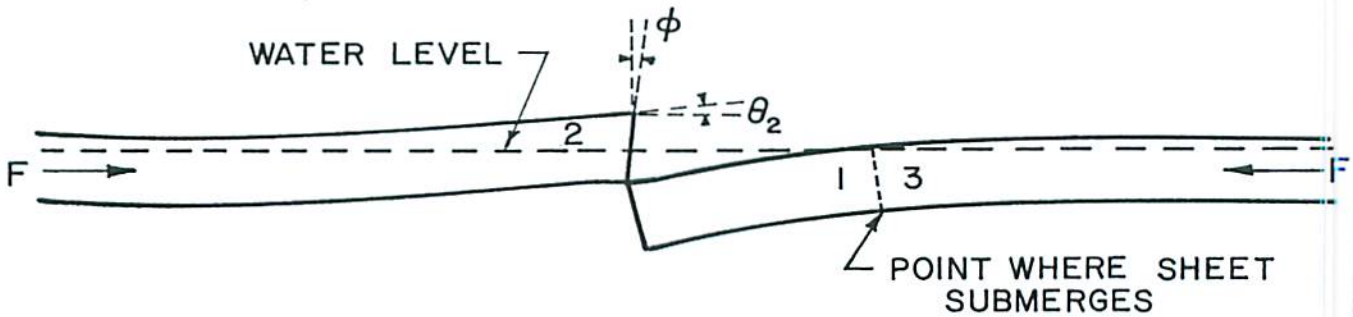

Figure 4

The edge of the sheet on the left is assumed to make an angle ϕ with the vertical to the surface of the ice. A frictionless interaction occurs between the two sheets (wet ice on ice). Figure 4 shows the two sheets in the configuration where rafting is about to occur. The force F and the maximum stress in the ice sheet can be calculated in this configuration. The interaction between the two sheets may be resolved into an axial force, a vertical force, and a moment.

The sheet on the right is partially submerged. The portion to the right of this point, labeled 3, is a beam on an elastic foundation with end loads. The submerged portion, labeled 1, is a beam column, without foundation, subjected to a constant distributed bouyancy force and end loads. The beam labeled 2 is not lifted from the water, and is therefore treated entirely as a beam on an elastic foundation with axial load and end loads. The angle which the edge of sheet 2 makes with the vertical to the water surface determines the relationship between F and the applied vertical force. This angle is $\phi - \theta_2$, where θ_2 is the rotation of the end of beam 2 due to the deformation. The rotation θ_2, and the

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length of beam 1 are important unknowns which must be eliminated during the solution of the problem.

The maximum bending stress is found to occur in the submerged beam. If this stress exceeds the strength of ice, pieces will be broken from the sheet and rafting will not occur. The broken pieces form rubble which then starts a ridge forming process [3]. The analytical results show that thin ice rafts, while thicker ice forms ridges. This conclusion has also been reached by Weeks and Kovacs [7] from field observations. They state that the transition from rafting to ridging usually occurs at a thickness between 15 cm and 30 cm, although on occasion thicker ice rafts without breaking.

The strength results can be presented in the form of equation (9). Vertical loads arise in two ways in this problem, at the interface due to the interaction of the two sheets, and as distributed moment resulting from the inplane force F acting through the vertical displacement. Rafting occurs when the total end displacement of the two sheets equals t. Thus, the vertical reactions of the foundation will be proportional to kt, while the shape function f will depend on F and the interactions. This suggests that we pick $q_0 = kt$ as a measure of load intensity.

The interactions in this problem consist of equal but opposite moments and concentrated forces at the interface. The magnitude of the vertical concentrated load is

$$P = F \tan(\phi - \theta_2)$$

and the moment is

$$M = \frac{Ft}{2}$$

The functional form of the loading can be specified by the ratio of these loads. To make a dimensionless ratio, note that P acts over the characteristic length $1/\lambda$ to create a moment of order P/λ. Thus the functional form of the load is given by the ratio

$$\frac{P}{\lambda M} = 2\,\frac{\tan(\phi - \theta_2)}{\lambda t}$$

We therefore introduce the dimensionless parameter

$$\psi = \frac{\tan(\phi - \theta_2)}{\lambda t}$$

The vertical reactions also depend on N, the inplane force parameter, and on the isostatic ratio $\xi = (\rho_w - \rho_i)/\rho_w$, which determines the freeboard of the ice, and thus the location where the overridden sheet submerges. For this case equation (9) takes the form

$$\frac{\sigma_c \lambda^2 t^2}{kt} = f_4(\psi, \xi, N)$$

In this problem, however, N is not an independent variable, since it is the force required to maintain equilibrium in a specified configuration, Figure 4. Thus, N in fact is a function of ψ and ξ. Therefore, the data may be presented in the form

$$\frac{\sigma_c \lambda^2 t}{k} = f_5(\psi, \xi)$$

The solution has been found, and the form of the function f_5 is shown in Figure 5

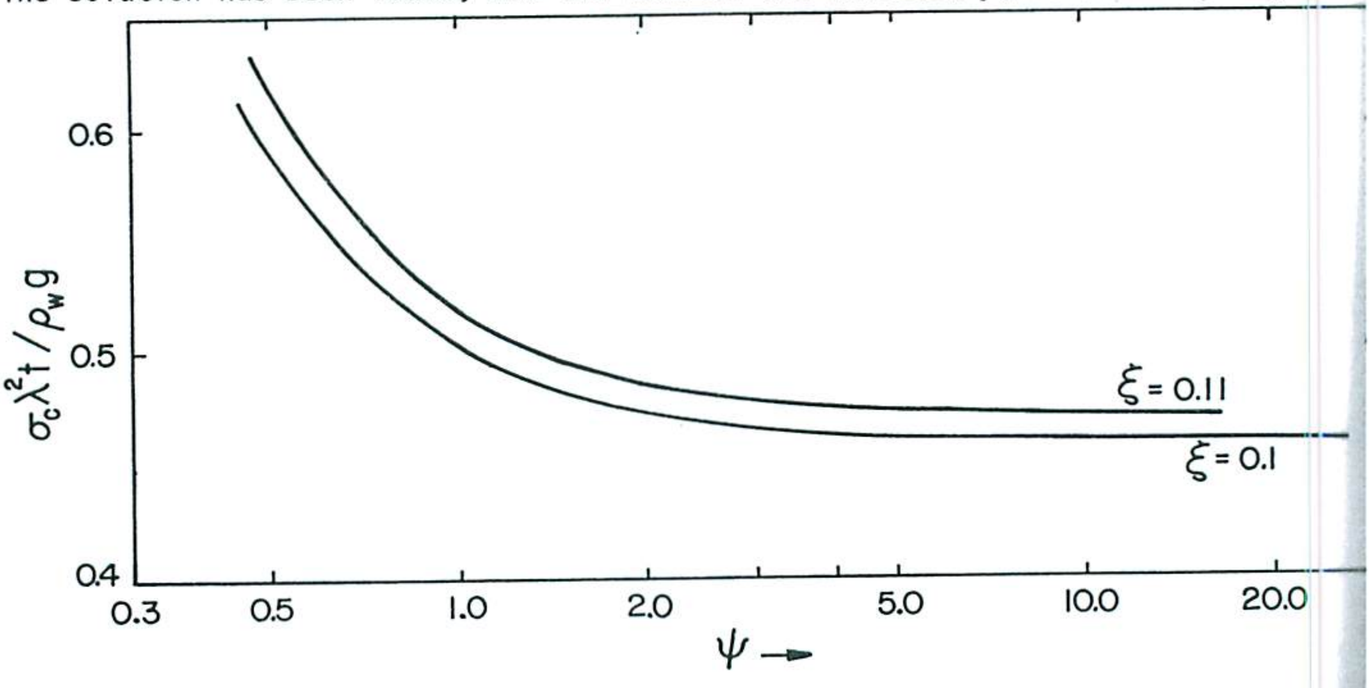

Figure 5

The maximum thickness of ice which will raft without failing can be calculated from Figure 5. For $\xi = 0.1$

$$\frac{\sigma_c \lambda^2 t}{\rho_w g} \simeq .46 \quad for\ \psi > 3.0$$

thus

$$\frac{\sigma_c t}{\rho_w g} \sqrt{\frac{3\rho_w(1 - \nu^2)g}{Et^3}} = .46$$

$$=> \qquad t = 14.2\ \frac{\sigma_c^2(1 - \nu^2)}{\rho_w g E} \qquad (10)$$

Note that the maximum thickness depends on the square of the strength σ_c. The strength and modulus of young ice are widely variable. Weeks and Anderson [8] have measured values of σ_c in the range $10^6 < \sigma_c < 3.5 \times 10^6$ $dyne/cm^2$. The measurement problem is compounded by the presence of a skeleton surface layer which is essentially without strength, so that the thickness of the load bearing ice is poorly defined. They also report a surprising amount of scatter in Young Modulus, in the range $10^9 < E < 8 \times 10^9$ $dynes/cm^2$. Assuming as nominal values $\sigma_c = 2 \times 10^6$ $dynes/cm^2$, $E = 3 \times 10^9$ $dynes/cm^2$, $\nu = 0.3$, $\rho_w g = 1009$ $dyne/cm^3$.

equation (10) yields

$$t = 17.1 \ cm$$

Thinner ice, with these properties, will tend to raft without breaking. Thicker ice will break during rafting, form rubble, and develop into ridge type structures. This result is in good agreement with the observations of Weeks and Kovacs [7]. The scatter in mechanical properties, and the strong dependence on σ_c easily account for the occasional rafting of abnormally thick ice.

CONCLUSIONS

Through dimensional analysis the variables which enter into problems of bending of the arctic ice can be conveniently grouped into a few dimensionless parameters. The strength parameters are identified and two example applications are studied. In one problem, the maximum height of ice which can pile against a sea wall is estimated using a simple model. The results are also applicable to ridge height calculations. In another example, the maximum thickness of ice which can raft is found to depend on the square of the strength.

These dimensionless parameters are a minimum set of variables for a given problem. They simplify the presentation of data, clarify the design of scale experiments, and reduce the numerical work required for analytical solutions.

Symbols not defined in text

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\,\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$$

$F_1 \cdots F_7$ functionals

$f_1 \cdots f_5$ functions

$$\delta_{ij} \begin{array}{ll} = 1 & i = j \\ = 0 & i \neq j \end{array}$$

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REFERENCES

1. Hetenyi M. 1946, *Beams on elastic foundation*, Ann Arbor, Michigan. Univ. of Michigan Press.
2. Timoshenko, 1940, *Theory of Plates and Shells*, McGraw-Hill.
3. Parmerter, R.R., and M.D. Coon 1973, "Mechanical models of ridging in the arctic sea ice cover," *AIDJEX Bulletin no. 19*, AIDJEX, Univ. of Washington, Seattle, Washington.
4. Parmerter, R.R., and M.D. Coon 1973, "On the mechanics of pressure ridge formation in sea ice," *Proceedings of the fifth Offshore Technology Conference*, Houston, Texas.
5. Parmerter, R.R. 1973, "A model of rafting" *AIDJEX Bulletin No. 23*, AIDJEX, University of Washington, Seattle, Washington.

6. Matsuoka, K 1972, "The mechanics of fracture of sea ice in leads." Master's Thesis, University of Washington.
7. Weeks, W.F. and A. Kovacs 1970, "On pressure ridges", report for Cold Regions Research and Engineering Laboratory, Hanover, New Hampshire.
8. Weeks, W.F. and D.L. Anderson 1958, "An experimental study of strength of young sea ice," *Transactions of the American Geophysical Union, Vol. 39, No. 4.*

DISCUSSION

Professor F. J. Legerer, Memorial University of Newfoundland, St. John's, Newfoundland, Canada:

Although I feel intrigued by the elegance of the treatment, I may submit that at this stage it is still incomplete; the following reason may be given to support my opinion: The results, for instance the height H_s of the rubble pile can only be measured statistically. Due to the nonlinear relationship of the independent variables and this result, the mean value of the result depends also on the higher order moments of the input if the stochastic aspects of the problem are considered. Hence, there must be a systematic difference between the statistical mean value, which can be measured and the value obtained from a deterministic analysis. Therefore, I suggest a stochastic extension of this very interesting note.

Professor R. Parmerter, AIDJEX, University of Washington, Seattle, Washington, U. S. A:

Your point is well taken. I agree that the statistical aspects of the problem would systematically modify the results. However, our goal was to develop a simple model which adequately represented the physical process. The additional complexity of a statistical approach did not seem justified at the time the model was developed. We felt that the simple model would show the correct dependence of height on critical parameters such as thickness and strenght. Perhaps the stoachastic aspects will be incorporated into a second generation model at some later date.