PORT AND OCEAN ENGINEERING UNDER ARCTIC CONDITIONS
TECHNICAL UNIVERSITY OF NORWAY

WIND-INDUCED CIRCULATION IN A LAKE.

Gunnar Lindh, Professor — Institute of Technology — Lund
Lars Bengtsson, Research Engineer — University of Lund — Sweden

The circulation in a lake is important among other things for prediction of diffusion of pollution.

The flow in a lake is mainly caused by wind, throughflow and density differences. Among these the wind-induced flow is believed to be the most important.

A mathematical model based on the common equations of motion and an assumption of hydrostatic pressure has been developed. The model considers the effects of Coriolis forces and can be applied on arbitrary lakes for arbitrary winds.

The efforts have been concentrated on the steady state. This might never be reached fully but should be useful as a description of the average situation. The non-linear terms are neglected which is possible according to prototype measurements. Numerical comparison of results obtained considering Coriolis forces with those obtained neglecting Coriolis forces shows that these forces should not be neglected even for small lakes. Parameters which must be known are the eddy-viscosity and the shear force caused by wind. The knowledge of these parameters is very limited especially for small lakes and for small wind velocities. The stratified lake is assumed to be divided into two homogeneous layers. The eddy-viscosity is assumed constant in each layer and in each vertical.

The model has been applied on lake Velen (A lake in central Sweden). This is a very irregular lake with an area of about 1 x 7 kms. After choosing an appropriate value of the shear force the eddy-viscosity was computed from the measured velocity profiles. Good correspondance was found between model and prototype.

To determine the influence of different parameters a simplified model, valid only in the middle of a lake, has been developed. This model may also be used for computing the eddy-viscosity.

Non-stationary flow has been studied only for the case of homogeneous lakes. This model requires knowledge of the bottom friction. The eddy-viscosity is allowed to vary between different levels. The horisontal turbulence is included in the model and so are the non-linear terms. This model employs almost exclusively differenttial methods.

The eddy-viscosity has a profound influence on the magnitude of flow, but has less influence on the circulation pattern. The bottom friction is negligible for deep lakes but is an important parameter for shallow lakes. The position of the pycnocline influences the magnitude as well as the direction of flow.

1. Introduction

The quality of the water of a lake is to a large extent dependent upon the circulation and diffusion processes in the lake. In order to judge the consequences of for instance different pollution discharges it is necessary to know the processes mentioned. Factors that cause circulation of the water masses in a lake include:

a) The wind. The shearing shess at the water surface excerted by the wind is usually the mechanism which decides the circulation within the lake.
b) The inflow-outflow system. This circulation is ordinarily of secondary importance but can be superimposed on the wind induced circulation.
c) Differences in the athmospheric pressure on the lake.
d) The astronomical forces. The factors c and d should not have any particular importance in lakes.
e) Density differences. These differences don't give rise to any particular circulation but modify the wind induced circulation.

In this paper the wind induced circulation of a lake is studied by using mathematical models. These models are compared to data obtained from field measurements.

2. Previous investigations

The equations that have been proposed in order to describe the circulationpattern are partly based on the analysis of oceanographic circulation phenomena, presented by Ekman (6) in the beginning of this century. In modern approaches to the problem of wind driven currents one ordinarily uses the vertically integrated equations of motion and introduces the streamfunctions. Svedrup and Stommel attacked the problem in this way and obtained solutions for rectangular ocean basin. In addition the contributions by Hidaka, Munk and Welander must be mentioned.

There are a surprisingly small number of papers regarding circulation in basins with smaller dimensions than the oceanic. Large-scale motions in a lake should be similar in character to the oceanic circulation. However, the currents in a lake are modified due to bottom friction and to the presence of coasts.

The introduction of high speed digital computer has made it possible to study more complete systems of equations. This is shown by Platzman (12), Ligget (9, 10, 11) and Csanady (1, 2, 3).

3. Theoretical investigations

As the wind blows over a lake it exerts a shearing stress at the water surface (τ). Momentum is transferred from the wind to the surface and by the turbulence transported further down into the water masses. The turbulence is introduced as vertical (ν_V) and horizontal (ν_H) eddy viscosity. When the water flows past the bottom of a lake, it is retarded due to the bottom friction (τ_B).

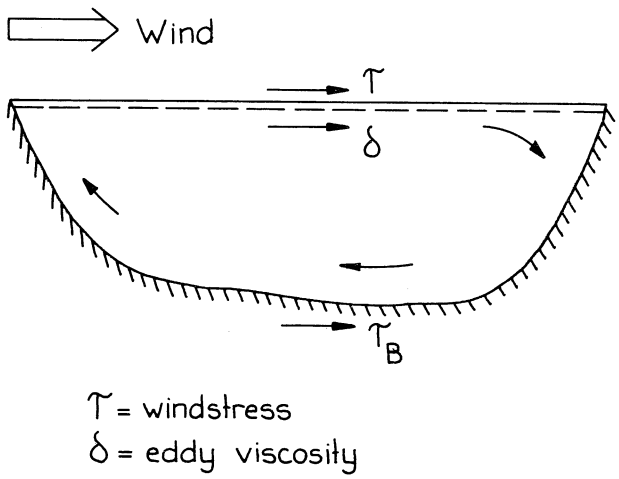


Fig 1. Wind induced circulation

The knowledge of the parameters τ and ν is very limited. So far the parameters of the model have been chosen to make the circulation fit the reality as far as possible.

The equations of motion of the turbulent circulation in a lake can be written

$$\frac{du}{dt} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\partial}{\partial z}(\nu_V \frac{\partial u}{\partial z}) + \frac{\partial}{\partial x}(\nu_{LO}\frac{\partial u}{\partial x}) + \frac{\partial}{\partial y}(\nu_{LA}\frac{\partial u}{\partial y}) \quad (1)$$

$$\frac{dv}{dt} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\partial}{\partial z}(\nu_V \frac{\partial v}{\partial z}) + \frac{\partial}{\partial x}(\nu_{LA}\frac{\partial v}{\partial x}) + \frac{\partial}{\partial y}(\nu_{LO}\frac{\partial v}{\partial y}) \quad (2)$$

where

u = velocity towards the east (x-direction)
v = velocity towards the north (y-direction)
w = velocity upwards (z-direction)
p = pressure
g = earth acceleration = 981 cm^2/s
f = Coriolis-parameter $10^{-4}/s$
τ = stress at the surface
τ_{xB}, τ_{yB} = stresses at the bottom
ν_V = vertical eddy viscosity
ν_{LO} = longitudinal eddy viscosity
ν_{LA} = lateral eddy viscosity
L = caracteristic length scale
D = depth scale

The second term on the left side is the Coriolis acceleration which must be added since the rotating earth isn't an inertial system. The terms on the right side are the pressure gradient and the turbulence terms. The vertical scale in a lake is much smaller than the horizontal one so the third equation of motion reduces to the hydrostatic pressure equation.

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = -g \quad (3)$$

The continuity equation for incompressible fluids is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (4)$$

The boundary conditions are given by $u = v = w = 0$ at all solid boundaries and known windstress (τ_x , τ_y) at the surface of the lake.

The Coriolis parameter, f, is regarded as a constant and the effect of waves and astronomical forces are not taken into account.

The equations are made dimensionless introducing

$$X^x = \frac{X}{L} ; \; Y^x = \frac{Y}{L} ; \; Z^x = \frac{Z}{D} ; \; U^x = \frac{fL}{gD} U ; \; V^x = \frac{fL}{gD} V ; \; W^x = \frac{fL^2}{gD^2} W ;$$

$$h^x = \frac{h}{D} ; \ p^x = \frac{p}{\rho g D} ; \ \tau^x = \frac{fL}{\nu g} \tau ; \ A_V = \frac{\nu_V}{fD^2} ; \ A_H = \frac{\nu_L}{fL^2} ; \ \nu_L = \nu_{LO} = \nu_{LA} =$$

$$= \text{constant} ; \ Ro = \text{Rossby number} = \frac{gD}{fL^2} ;$$

$$\tau^x = \frac{\partial u}{\partial z} ; \ t^x = ft ;$$

By integrating the continuity equation the circulation will be non divergent. Thus a stream function can be defined as

$$h\bar{u} = \frac{\partial \psi}{\partial y} \qquad h\bar{v} = - \frac{\partial \psi}{\partial x} \tag{5}$$

where the bar indicates the vertically averaged velocity and h the depth.

The equations of motion are now being integrated vertically over a homogenous layer, then divided by the depth h. Eq. (1) is derivated with respect to y and eq. (2) with respect to x and finally subtracted from each other. In this way the pressure terms will be eliminated. The result is the differential equation

$$\frac{\partial}{\partial t} (\nabla^2 \psi - \frac{1}{h} (\frac{\partial \psi}{\partial y} \frac{\partial h}{\partial y} + \frac{\partial \psi}{\partial x} \frac{\partial h}{\partial x})) + R_o \cdot K =$$

$$- \frac{1}{h} (\frac{\partial \psi}{\partial y} \frac{\partial h}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial h}{\partial y}) + \frac{\partial}{\partial y} (A_V \tau_x - A_V \tau_{xB}) + \frac{\partial}{\partial x} (A_V \tau_y - A_V \tau_{yB}) -$$

$$- \frac{1}{h} (\frac{\partial h}{\partial y} (A_V \tau_x - A_V \tau_{xB}) - \frac{\partial h}{\partial x} (A_V \tau_y - A_V \tau_{yB})) +$$

$$+ A_H \cdot \{\nabla^4 \psi - \frac{1}{h} (\frac{\partial h}{\partial y} (\frac{\partial^3}{\partial x^2 \partial y} \psi + \frac{\partial^3}{\partial y^3} \psi) + \frac{h}{\partial x} (\frac{\partial^3}{\partial x^3} \psi + \frac{\partial^3 \psi}{\partial y^2 \partial x})\}$$

where

$$K = (\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2}) \overline{huv} + \frac{\partial^2}{\partial x \partial y} (\overline{hu^2} - \overline{hv^2}) \tag{6}$$

and the boundary conditions are $\psi = 0$ and $\frac{\partial \psi}{\partial n} = 0$.

4. Exchange phenomena

Due to the turbulent exchange of momentum it is not possible to formulate uniquely an expression for the acceleration induced by viscosity. By introducing the concept of eddy viscosity one gets an analogy to the molecular viscosity. The coefficient is not a characteristic property of the fluid but it is anisotropic and depends on the flow. Momentum is transferred from the wind to the water surface. The relationship between the wind and the shearing force is commonly assumed to be

$$\tau = \rho \cdot C_D \cdot W^2 \tag{7}$$

where ρ is the density of the air, C_D is a dragcoefficient and W is the wind velocity. However C_D is a function of the strength of the wind, the dimensions of the lake and the stratification of the air. Other wind profiles such as the Deacon profile (5) or van Kármán flow have also been used.

5. Mathematical models

5.1. Model A

The first model A is based on eq. (6) where the lake is assumed to be barotropic. The windstress and its distribution, the eddy viscosity which can vary from one point to another in three dimensions and the configuration of the lake are read into the computer. The variables are made dimensionless. For every time step of two minutes the stream functions are solved from eq. (6). For the long time steps, first the pressure gradients from the integrated equations of motion are solved then the stream velocities are solved. The bottom friction is assumed to be a constant times the bottom velocity. So the step between the pressure gradients and the stream velocities must be repeated about three times.

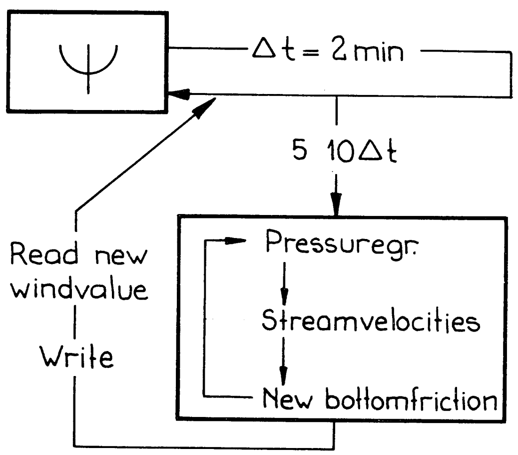


Fig 2. Iteration scheme for model A

This model demands a large computer memory and a long computer time so therefore it is only used to show the importance of different factors.

5.2. Model B (cf. Fig 3)

In model B the stream velocities are considered non-accelerated. The vertical eddy viscosity is regarded as constant throughout the vertical, the horizontal

Model B

1. The stress, eddy viscosity and bottom-configuration are read into the computer.
2. The variables are made dimensionless.
3. The streamfunctions are solved by iteration of (10).
4. The pressure gradients are solved.
5. The constants of 8-9 are solved.
6. The streamvelocities are solved from 8-9.

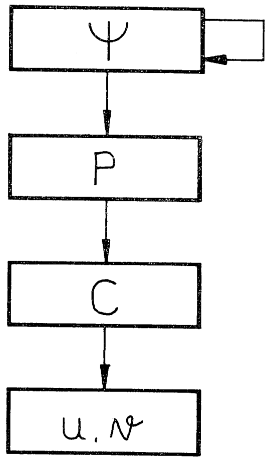


Fig 3.

turbulence is neglected and the water masses are assumed to be homogenous. In dimensionless form the velocities can be expressed as

$$u = -\frac{\partial p}{\partial y} + \cos Kz\,(C_1 e^{Kz} + C_2 e^{-Kz}) + \sin Kz\,(C_3 e^{Kz} + C_4 e^{-Kz}) \qquad (8)$$

$$v = \frac{\partial p}{\partial x} + \cos Kz\,(-C_3 e^{Kz} + C_4 e^{-Kz}) + \sin Kz\,(C_1 e^{Kz} - C_2 e^{-Kz}) \qquad (9)$$

where $K = (A_v/2)^{1/2}$ and the "constants" C_1 to C_4 are functions of x and y.

The boundary conditions are u = v = 0 at the bottom. Moreover τ_x and τ_y must be known at the surface. The constants are different at every point of the lake but they can be determined since the velocity disappears at the bottom and in addition the windstress at the surface is known. However, they are connected to the pressure gradients. The streamfunctions are computed by iteration of the differential equation which is the result of integration and derivation of eqs. (8) and (9). Thus

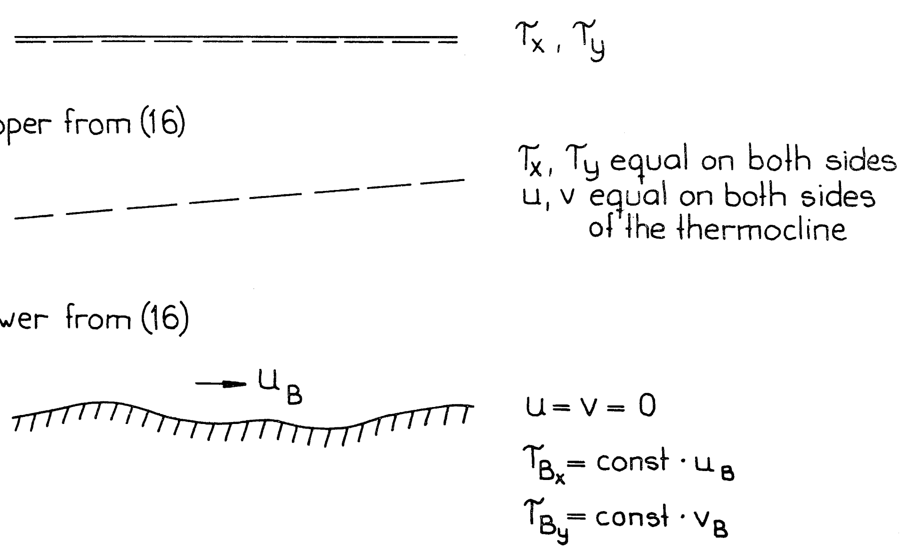
$$\nabla^2 \psi + A\frac{\partial \psi}{\partial x} + B\frac{\partial \psi}{\partial y} + C\,\tau_x + D\,\tau_y + E\left(\frac{\partial \tau_x}{\partial x} + \frac{\partial \tau_y}{\partial y}\right) + F\left(\frac{\partial \tau_x}{\partial y} - \frac{\partial \tau_y}{\partial x}\right) = 0 \qquad (10)$$

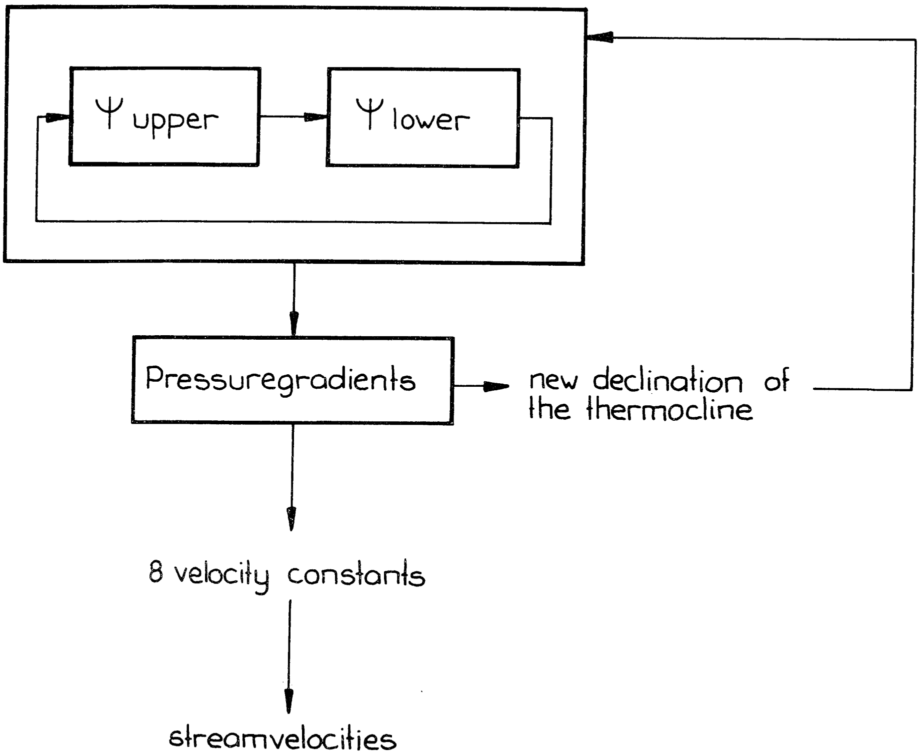
where A to F are functions of x and **y.**

5.3. Model C

Model C is very similar to model B but the water masses are now stratified in two layers. Within each layer the density and the eddy viscosity are considered constant. The solutions of the stream velocities will be as in model B, eqs. (8) and (9), but the "constants" will be different in the two layers. As both the velocities and the stresses must be equal on both sides of the discontinuity surface **all** eight constants can be determined. In this case there will be two partial differential equations since the stream functions are different in the two layers. First the stream functions are determined then the pressure gradients and the constants. Finally the stream velocities are very easily solved. The model C is shown in Fig 4.

Model C





5.4. Model D

The three models mentioned all demand rather large computer programs. In order to study only the effect of different phenomena such as for example the depth of the thermocline, the fourth model D is used. The equations for the stream velocities will be the same as in model C. As only one vertical is studied one needs the boundary conditions from model C and the demand of continuity in the vertical within each layer.

6. Experiences of the models

6.1. Model A

Due to the long computer time only a few runs have continued until stationarity has been reached. Fig 5-7 show the surface current as function of time. One can notice that for water masses 0,2 - 0,5 km^3 stationarity is reached after about three hours. Along the coasts stationarity is reached within one hour.

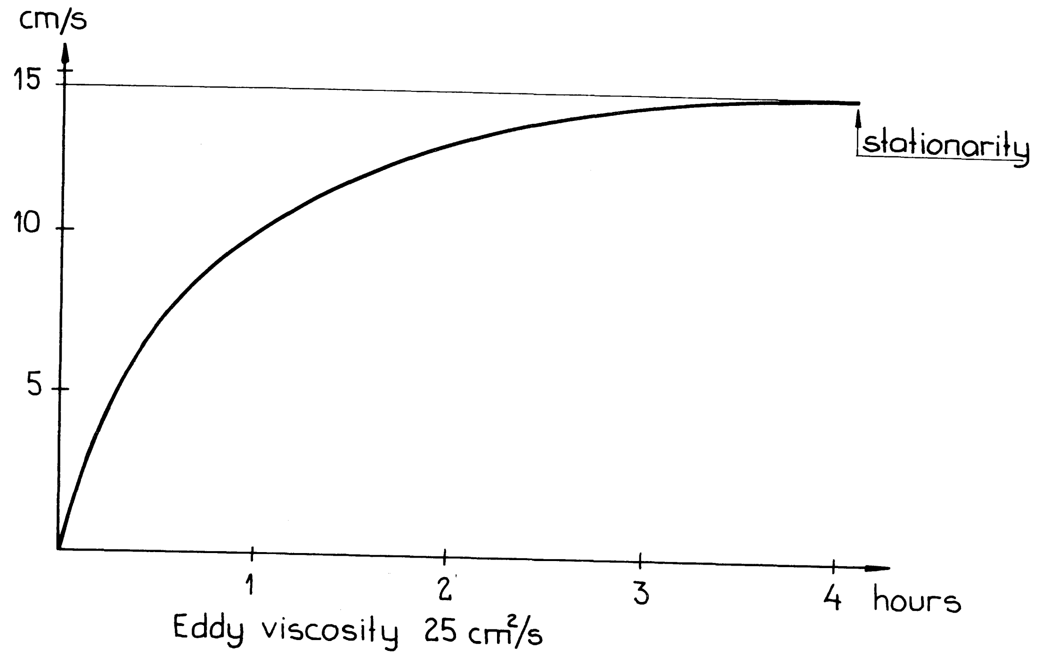


Fig 5.

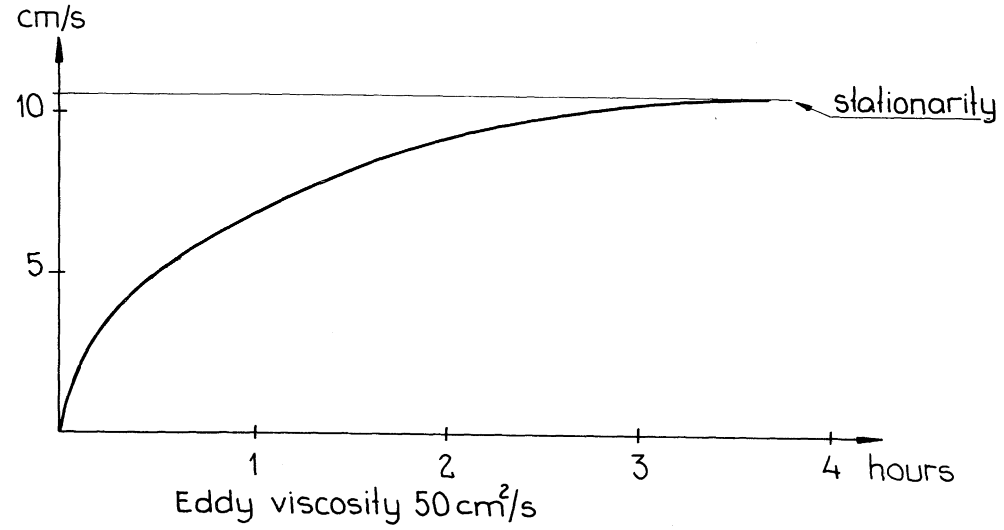


Fig 6.

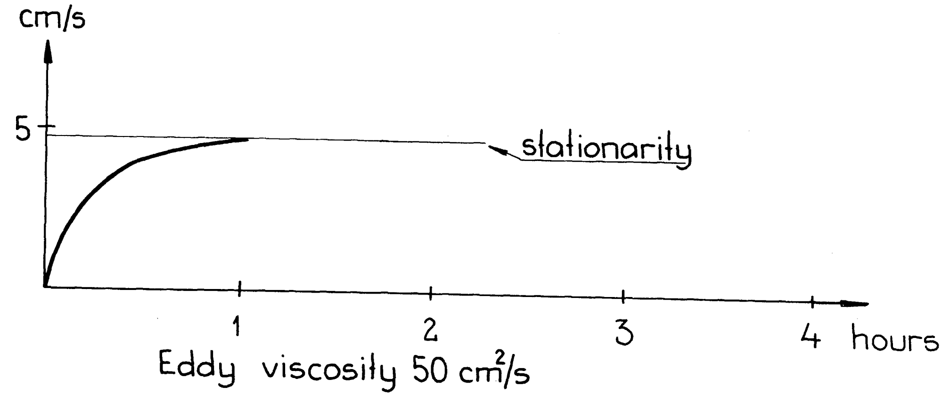


Fig 7.

Fig 8 shows how the velocity profile gradually develops. In this case the bottom friction has been disregarded. Therefore the profile at the bottom is very sharp.

This model can take into account the effect of horizontal turbulence, but this has very little influence on the circulation pattern.

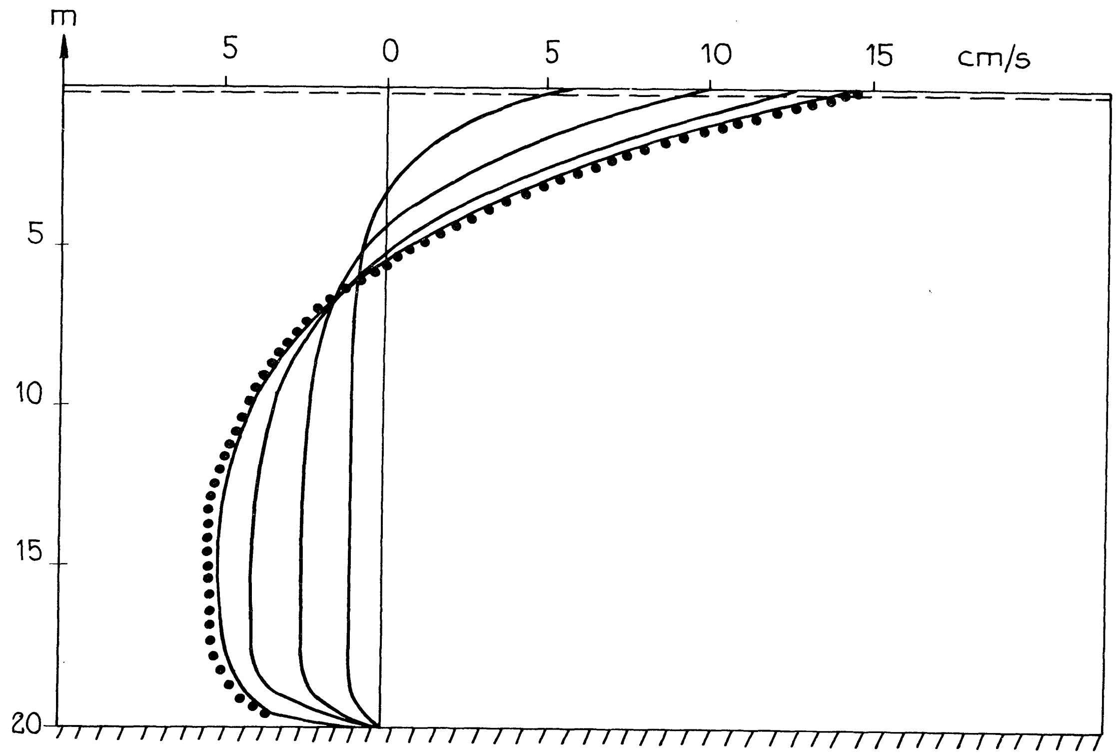


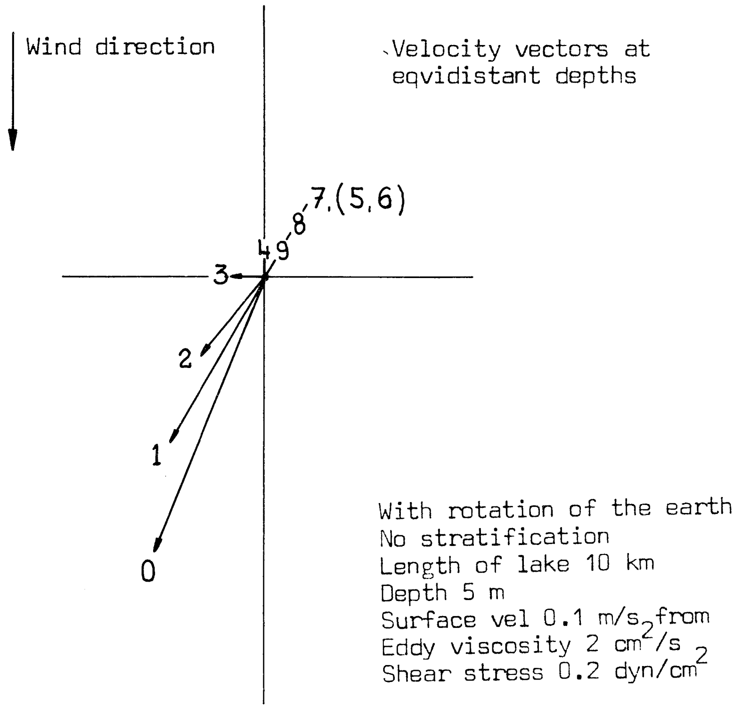
Fig 8. Velocity profile after 20 min, 1, 2, 3 hours and stationarity (dotted line)

6.2. Model B

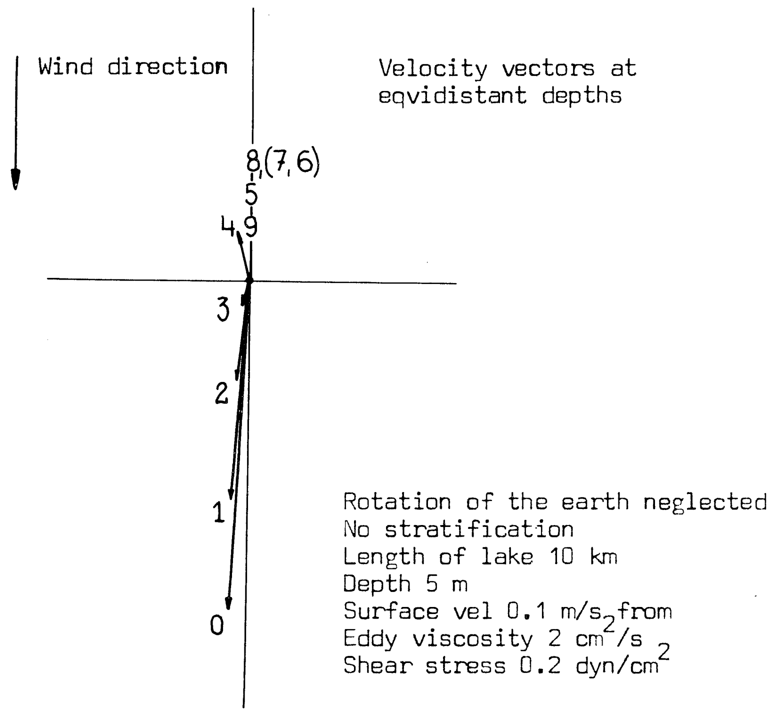
If model B is studied with or without earth rotation it appears that not even for small lakes the earth rotation can be disregarded, cf. Fig 9 and 10.

From the first Figure one can see how the earth rotation declines the current to the right. The currents decrease when the depth increase. The circulation in these Figures have been reached with a plane bottom and a low value of the eddy viscosity otherwise the Coriolis effect would have been less. The effect of a small vertical eddy viscosity can be summarized as follows

a) a strong surface current but the currents die out fast when the depth of water increases
b) compared to the central part of the lake strong currents develop along the coasts
c) a return current near the surface
d) no return current along the coasts
e) a strong influence caused by the earth rotation



Wind direction
Velocity vectors at eqvidistant depths
7,(5,6)
8
4
9
3
2
1
0
With rotation of the earth
No stratification
Length of lake 10 km
Depth 5 m
Surface vel 0.1 m/s from
Eddy viscosity 2 cm2/s
Shear stress 0.2 dyn/cm2



Wind direction
Velocity vectors at eqvidistant depths
8,(7,6)
5
4,9
3
2
1
0
Rotation of the earth neglected
No stratification
Length of lake 10 km
Depth 5 m
Surface vel 0.1 m/s from
Eddy viscosity 2 cm2/s
Shear stress 0.2 dyn/cm2

6.3. Experiences of the stratified models

A typical profile from one of the stratified models is shown in Figure 11.

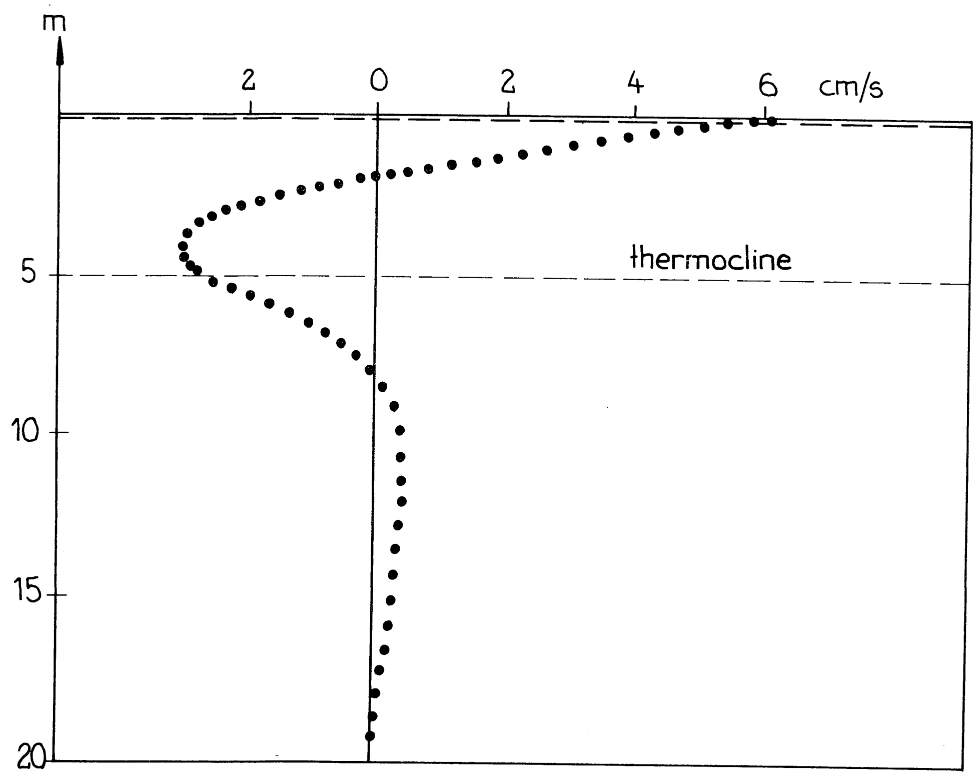


Fig 11.

One may observe how the stress has a minimum value just above the thermocline, which also agree with field observations.

The depth of the thermocline has a strong influence on the circulation. An increase of the depth of the thermocline gives rise to

a) a stronger surface current

b) stronger lateral velocities that is a stronger influence from the earth rotation

7. A comparision between model and reality

Model B has been applied to Lake Velen in Sweden. This is a very irregular lake with an area of about 1 km x 7 km, Fig 12. The current measurements were made with drogues. The water masses were almost homogenous. The wind was WSW 5 m/s. The best correspondence between theory and observations was found when τ = 0,35 dyn/cm^2 and ν = 15 cm^2/s. For the larger part of the lake the comparison could only be made down to about 5 m as the drogues otherwise very soon touched the bottom. A comparision in the deeper part of the lake is shown in Figure 12. Dotted lines are calculated and full lines are measured stream velocities.

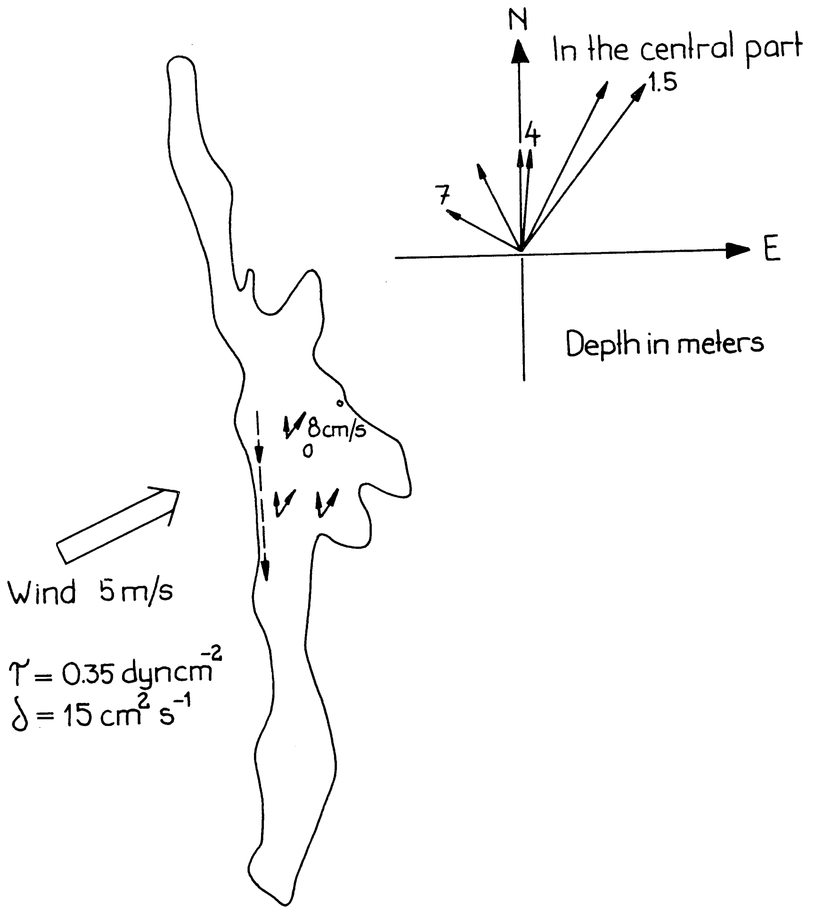


Fig 12.

The theoretically calculated circulation pattern has shown

a) a surfacecurrent of about 8 cm/s towards NNE

b) a current turning to the left with increasing depth

c) a southern return current for the whole lake at the depth of 9 m however along the western coast already appearing at the depth of 5 m

Observations from the whole lake can be summarized in the following manner

a) in reality the surface currents were almost the same as the calculated ones

b) in reality there was also a left turning current

c) there was a shallow return current along the western coast

8. Conclusions

Due to the lack of incomplete knowledge of the drag coefficient and the eddy viscosity it seems reasonable to use only a linear model. Model A is too intractable to work with. Iterations in three dimensions are time consuming which also means an increasing uncertainty. Further studies will be devoted to development of the analytically more advanced models B and C. Further knowledge of the eddy viscosity and the windstress is needed. The horizontal turbulence can eventually be taken into account in the form of the Guldborg-Mohn assumption.

These models can be used for many practical purposes. As an example they can be used to predict how lake reacts as a receiver of waste water.

Litterature

1. Csanady, G T: Wind-Driven Summer Circulation in the Great Lakes. J of Geoph Res, aug 1967.
2. Csanady, G T: Turbulence and Diffusion in the Great Lakes. - Great Lakes Res. Div., Publ No 11, 1964.
3. Csanady, G T: Turbulent Diffusion in Lake Huron. - J Fluid Mech, 17, 1963.
4. Deacon, E L: Vertical Diffusion in the lowest layers of the atmosphere. - J Roy Mel Soc 49.
5. Deacon, E L: Aerodynamic Roughness of the Sea. - J Geoph Res, july 1962.
6. Ekman, V: On the Influence of the Earth's Rotation on Ocean-Currents. - Arkiv Mat Astronomi, Fysik. Band 2, Nr 11, 1905.
7. Engelund, F: Dispersion of floating particles in uniform channel. - J Hydr Div, july 1969.
8. Ichie, T: A note on the Friction Terms in the Equations of Ocean Currents.
9. Ligget, I A och Hadjitheoudorou, Ch: Circulation in Shallow Homogenous Lakes. - J Hydr Div, mars 1969.
10. Ligget, J A: Unsteady Circulation in Shallow Homogenous Lakes. - J Hydr Div, July 1969.
11. Ligget, J A: Cell Method for Computing Lake Circulation. - J Hydr Div, mars 1970.
12. Platzman, G W: Numerical Computation of Wind Tides on Lake Erie. - Great Lakes Res D V, pub 7.
13. Taveycev, B A: Non geostrophic Disturbance and Baroclinie Instability in Two-Layered Oceanic Flow.